Bachelor of Science

(B.Sc.)

INTRODUCTORY PHYSICS

Semester-I

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UNIT 1: VECTOR

STRUCTURE:

1.0 Objective:

After reading this unit you will be able to understand:
* Defining vector.
*Vector representation, addition, subtraction
*Orthogonal representation
*Multiplication of vectors
*Scalar product, vector product
*Scalar triple product and vector triple product

1.1 Introduction:

On the basis of direction, the physical quantities may be divided into two main classes.

1.1.1 Scalar quantities: The physical quantities which do not require direction for their representation. These quantities require only magnitude and unit and are added according to the usual rules of algebra. Examples of these quantities are: mass, length, area, volume, distance, time speed, density, electric current, temperature, work etc.

1.1.2 Vector quantities: The physical quantities which require both magnitude and direction and which can be added according to the vector laws of addition are called vector quantities or vector. These quantities require magnitude, unit and direction. Examples are weight, displacement, velocity, acceleration, magnetic field, current density, electric field, momentum angular velocity, force etc.

1.2 Vector representation:

Any vector quantity say A, is represented by putting a small arrow above the physical quantity like $\vec{A}$. In case of print text a vector quantity is represented by bold type letter like **A**. The vector can be represented by both capital and small letters. The magnitude of a vector quantity A is denoted by $|\vec{A}|$ or *mod A* or some time light forced italic letter *A*. We should understand following types of vectors and their representations.

1.2.1 Unit vector

A unit vector of any vector quantity is that vector which has unit magnitude. Suppose $\vec{A}\, is\ a\ vector$ then unit vector is defined as

$$\hat{A} = \frac{\vec{A}}{|A|}$$

The unit vector is denoted by $\hat{A}$ and read as 'A *unit vector or A hat'*. It is clear that the magnitude of unit vector is always 1. A unit vector merely indicates direction only. In Cartesian coordinate system, the unit vector along x, y and x axis are represented by $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ respectively as shown in figure 1.1.

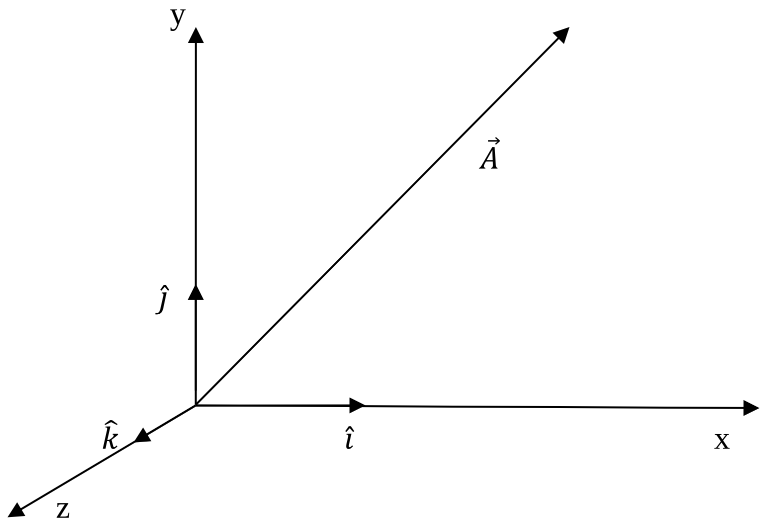


Figure 1.1

Any vector in Cartesian coordinate system can be represented as

$$A = \hat{\imath}\, A_x + \hat{\jmath}\, A_y + \hat{k}\, A_z$$

Where $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ are unit vector along x, y z axis and, A_x , A_y , A_z are the magnitudes projections or components of $\vec{A}$ along x, y, z axis respectively.

The unit vector in Cartesian coordinate system can be given as:

$$\hat{A} = \frac{\hat{\imath}\, A_x + \hat{\jmath}\, A_y + \hat{k}\, A_z}{\sqrt{A_x{}^2 + B_x{}^2 + C_x{}^2}}$$

1.2.2. Zero vector or Null vector:

A vector with zero magnitude is called zero vector or null vector. The condition for null vector is $|\vec{A}| = 0$

1.2.3 Equal vectors:

If two vectors have same magnitude and same direction, the vectors are called equal vector.

1.2.4 Like vectors:

If two or more vectors have same direction, but may have different magnitude, then the vectors are called like vectors.

1.2.5 Negative vector:
A vector is called negative vector with reference to another one, if both have same magnitude but opposite directions.

1.2.6 Collinear vectors:
All the vectors parallel to each other are called collinear vectors. Basically collinear means the line of action is along the same line.

1.2.6 Coplanar vector:
All the vectors whose line of action lies on a same plane are called coplanar vectors. Basically coplanar means lies on the same plane.

1.2.7. Graphical representation of vectors:
Graphically a vector quantity is represented by an arrow shaped straight line, with suitable length which represents magnitude, and the direction of arrow represents direction of vector quantity. For example, if a force $\vec{A}$ is directed towards east and another force $\vec{B}$ is directed toward north-west then these forces can be represented as shown in figure 1.2.

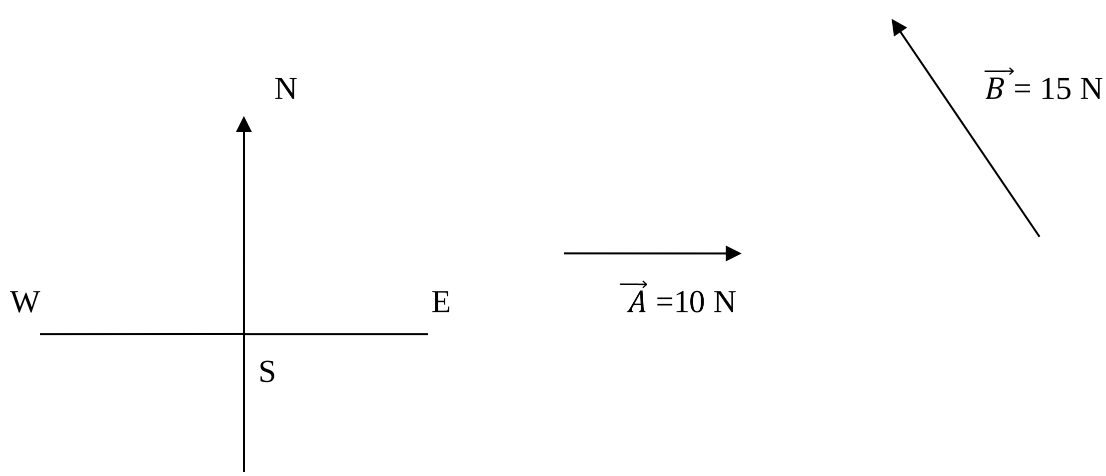


Figure 1.2

1.2.8 Addition and subtraction of vectors:
The addition of two vectors can be performed by following two laws.

(A) The parallelogram law:

According to this law, if two vectors $\vec{A}$ and $\vec{B}$ are represented by two adjacent sides of a parallelogram as show in figure 1.3, then the sum of these two vectors or resultant $\vec{R}$ is represented by the diagonal of Parallelogram.

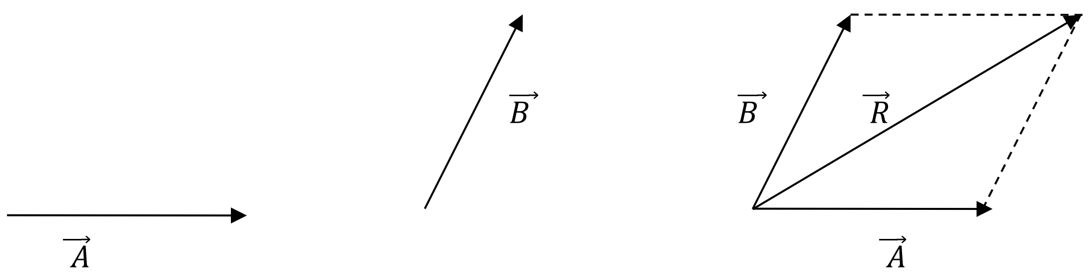


Figure 1.3

If vector $\vec{A}$ and $\vec{B}$ are represented by the sides of a parallelogram as shown in figure 1.4 and the angle between $\vec{A}$ and $\vec{B}$ is θ, and resultant $\vec{R}$ makes angle α with vector $\vec{A}$ then magnitude of $\vec{R}$ is

$$|R| = \sqrt{A^2 + B^2 + 2AB\cos\theta}$$

The angle α is given as

$$\alpha = tan^{-1}\frac{B\sin\theta}{A + Bcos\,\theta}$$

You should notice that all three vectors $\vec{A}$, $\vec{B}$ and $\vec{R}$ are concurrent *i.e*. vectors acting on the same point O.

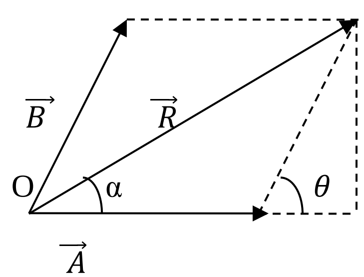


Figure 1.4

(B) Triangle law:

According to this law if a vector is placed at the head of another vector, and these two vectors represent two sides of a triangle then the third side or a vector drawn for the tail end of first to the head end of second represents the resultant of these two vectors. If vectors $\vec{A}$ and $\vec{B}$ are two vector as shown in figure 1.5, then resultant $\vec{R}$ can be obtained by applying triangle law.

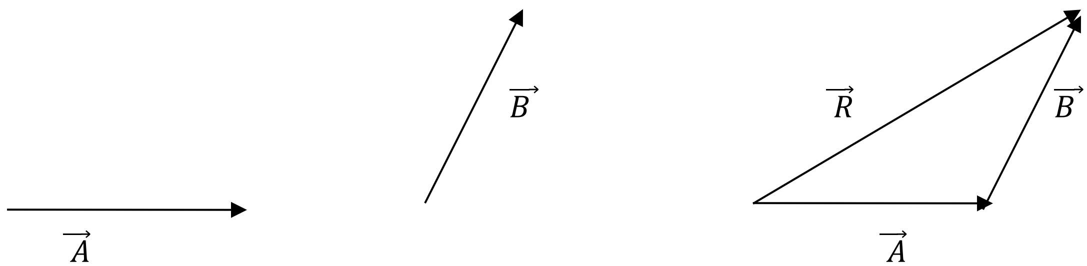


Figure 1.5

(c) Polygon law of vector addition:

This law is used for the addition of more than two vectors. According to this law if we have a large number of vectors, place the tail end of each successive vector at the head end of previous one. The resultant of all vectors can be obtained by drawing a vector from the tail end of first to the head end of the last vector. Figure 1.6 shows the polynomial law of vector addition different vectors $\vec{A}$,$\vec{B}$, $\vec{C}$, $\vec{D}$, $\vec{E}$ etc. and $\vec{R}$ is resultant vector.

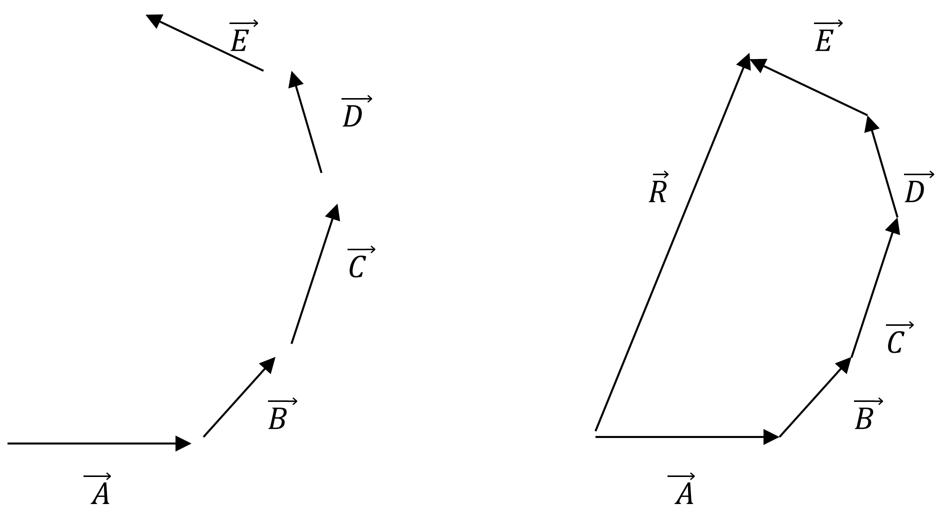


Figure 1.6

1.2.9 Resolution of vector:

A vector can be resolved into two or more vectors and these vectors can be added in accordance with the polygon law of vector addition, and finally original vector can be obtained. If a vector is resolved into three components which are mutually perpendicular to each other then these are called rectangular components or mutual perpendicular components of a vector. These components are along the three coordinate axes x, y and z respectively as show in figure 1.7.

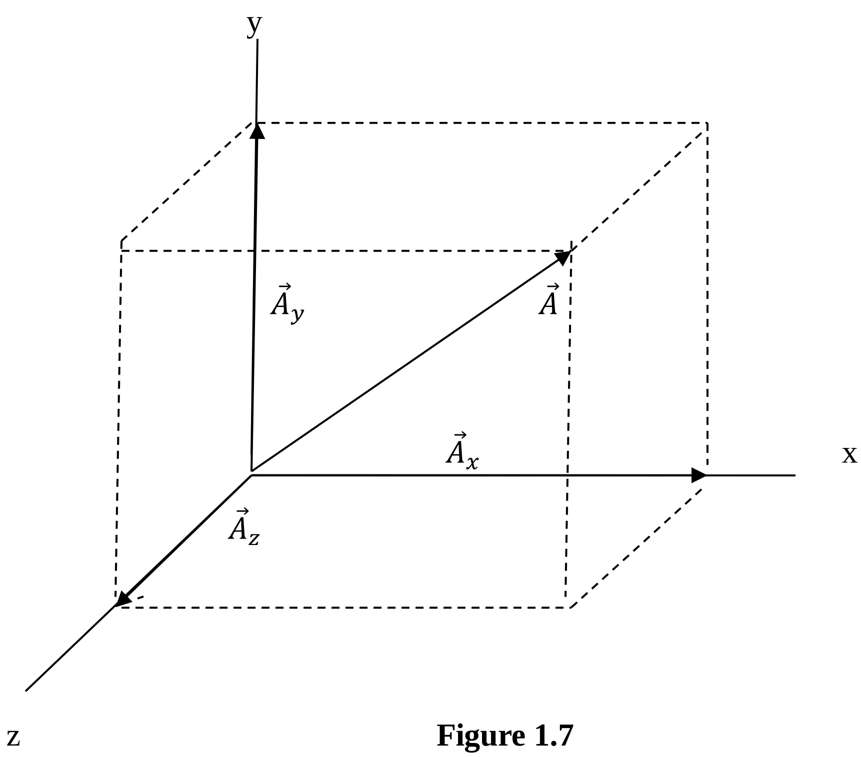


Figure 1.7

If the unit vectors along x, y and x axis are represented by $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ respectively then any vector $\vec{A}$ can be give as

$$\vec{A} = \hat{\imath}\, \mathrm{A_x} + \hat{\jmath}\, \mathrm{A_y} + \hat{k}\, \mathrm{A_z}$$

$\vec{A}$ constitutes the diagonal of a parallelepiped, and $\mathrm{A_x}$, $\mathrm{A_y}$ and $\mathrm{A_z}$ are the edges along x, y and z axes respectively. $\vec{A}$ is polynomial addition of vectors $\mathrm{A_x}$, $\mathrm{A_y}$ and $\mathrm{A_z}$. The rectangular components $\mathrm{A_x}$, $\mathrm{A_y}$ and $\mathrm{A_z}$ can be considered as orthogonal projections of vector $\vec{A}$ on x, y and z axis respectively. Mathematically, the magnitude of vector $\vec{A}$ can be given as:

$$\boldsymbol{A} = |\vec{A}| = \sqrt{A_x{}^2 + A_y{}^2 + A_z{}^2}$$

1.2.10 Direction cosines:

The cosine of angles which the vector $\vec{A}$ makes with three mutual perpendicular axes x, y and z are called direction cosine and generally represented by l, m, n respectively. In figure 1.8 vector $\vec{A}$ makes angle α, β *and* γ with axis x, y and z respectively. Then

$$\cos\alpha = \frac{A_x}{A} = \frac{A_x}{\sqrt{A_x{}^2 + A_y{}^2 + A_z{}^2}}\,; \quad \cos\beta = \frac{A_y}{A} = \frac{A_y}{\sqrt{A_x{}^2 + A_y{}^2 + A_z{}^2}}\,;$$

$$\cos\gamma = \frac{A_z}{A} = \frac{A_z}{\sqrt{A_x{}^2 + A_y{}^2 + A_z{}^2}}$$

Where A_x , A_y and A_z are the projection or intercepts of vector $\vec{A}$ along x, y and z axes respectively. The $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called direction cosines.

$$l = \cos\alpha; \quad m = \cos\beta\,; n = \cos\gamma$$

Mathematically

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ or $\quad l^2 + m^2 + n^2 = 1$

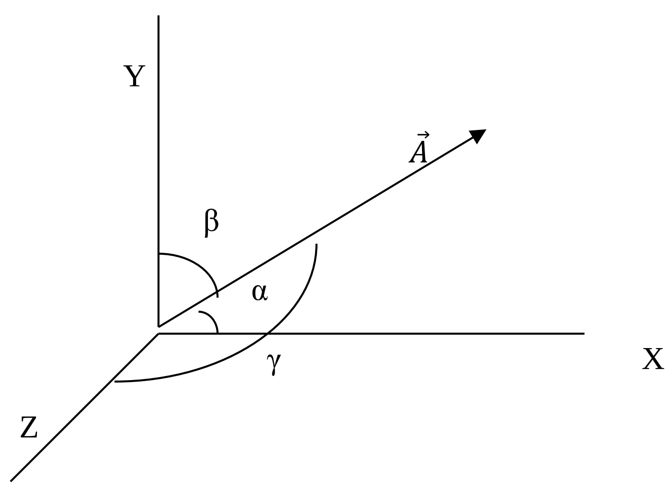


Figure 1.8

1.2.11 Position vector:

In Cartesian co-ordinate system the position of any point P(x, y, z) can be represented by a vector **r**, with respect to origin O then the vector **r** is called position vector of point P. Position vector is often denoted by $\bar{r}$. Figure1.9 shows the position vector of a point P in Cartesian coordinate system. If we have two vectors $\vec{P}$ and $\vec{Q}$ with position vectors $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ respectively then

$$\boldsymbol{r_1} = \hat{\imath}\, x_1 + \hat{\jmath}\, y_1 + \hat{k}\, z_1$$

$$\boldsymbol{r_2} = \hat{\imath}\, x_2 + \hat{\jmath}\, y_2 + \hat{k}\, z_2$$

Where (x_1, y_1, z_1) and (x_2, y_2, z_2) are the coordinates of point P and Q respectively.

Now the vector PQ can be given as

PQ = OQ – OP ($\therefore$ OP + PQ = OQ)

$$\bar{r} = \bar{r_2} - \bar{r_1}$$

Therefore, vector PQ = position vector of Q − position vector of P

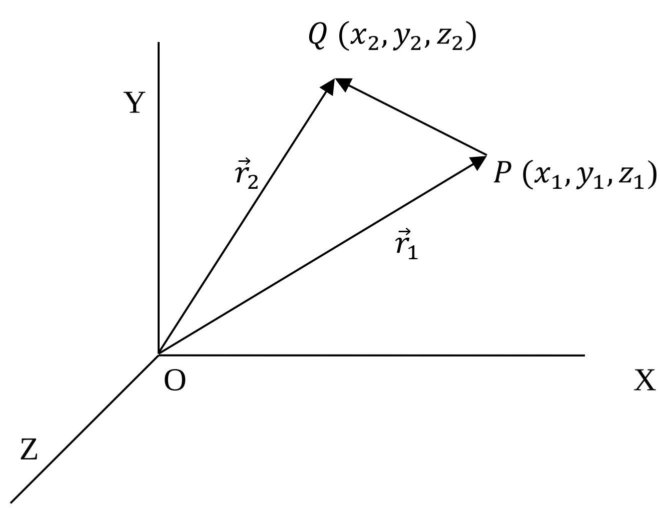


Figure1.9

1.3 Multiplication of vectors:

1.3.1 Multiplication and division of a vector by scalar:

If a vector **P** is multiplied by a scalar quantity m then its magnitude becomes m times. For example if m is a scalar and $\vec{A}$ is a vector then its magnitude becomes m times.
Similarly, in case of division of a vector A by a non zero scalar quantity n, its magnitude becomes 1/n times.

1.3.2 Product of two vectors:

There are two distinct ways in which we can define the product of two vectors.

1.3.2.1 Scalar product or dot product:

Scalar product of two vectors **P** and **Q** is defined as the product of magnitude of two vectors P and Q and cosine of the angle between the directions of these vectors.

If θ is the angle between two vectors $\vec{P}$ and $\vec{Q}$, then dot product (*read as* $\vec{P}$ *dot* $\vec{Q}$) of two vectors is given by-

$$\overrightarrow{P}.\ \vec{Q}\ = PQ\cos\theta\ = P\ (Q\cos\theta)$$

$$= P\ (projection\ of\ vector\ Q\ on\ P) =\ P.MN$$

The figure 1.10 shows the dot product. The resultant of dot product or scalar product of two vectors is always a scalar quantity. In physics the dot product is frequently used, the simplest example is work which is dot product of force and displacement vectors.

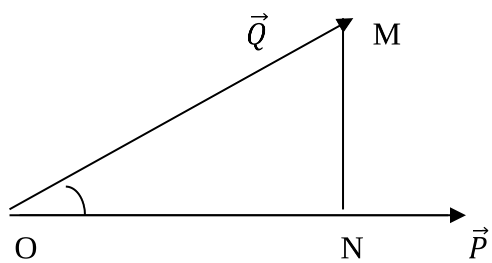


Figure1.10

Important properties of dot product

(i) Condition for two collinear vectors:
If two vectors are parallel or angle between two vectors is 0 or π, then vectors are called collinear. In this case

$$\overrightarrow{P}.\ \vec{Q}\ = PQcos\,0^o\ = PQ$$

Then the product of two vectors is same as the product of their magnitudes.

(ii) Condition for two vector to be perpendicular to each other:

If two vectors are perpendicular to each other then the angle between these two vectors is 90^0, then

$$\overrightarrow{P}.\vec{Q} = PQcos\,90^o = 0$$

Hence two vectors are perpendicular to each other if and only if their dot product is zero.

In case of unit vectors $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ we know that these vectors are perpendicular to each other then

$\hat{\imath}\,.\,\hat{\jmath} = \hat{\imath}.\,\hat{k} = \hat{k}.\hat{\imath} = 0$

similarly

$\hat{\imath}\,.\,\hat{\imath} = \hat{\jmath}.\,\hat{\jmath} = \hat{k}.\,\hat{k} = 1$

(iii) Commutative law holds:

In case of vector dot product the commutative law holds. Then

$$\overrightarrow{P}.\vec{Q} = \vec{Q}.\vec{P}$$

(iv) Distributive property of scalar product:

If P, Q and R are three vectors then according to distributive law

$\vec{P}.(\vec{Q}+\vec{R}) = \vec{P}.\vec{Q}+\vec{P}.\vec{R}$

Example 1.1 Show that vector $\vec{A} = 3i + 6j - 2k$ and $\vec{B} = 4i - \hat{\jmath} + 3k$ are mutually perpendicular.

Solution: If the angle between $\vec{A}$ and $\vec{B}$ is θ then

$\vec{A}\,.\,\vec{B} = A\,B\cos\theta$

$\cos\theta = \frac{\vec{A}.\vec{B}}{AB} = \frac{(3i+6j-2k).(4i-j+3k)}{\sqrt{(A_x^2+A_y^2+A_z^2)}\sqrt{(B_x^2+B_y^2+B_z^2}} = 0$

$\cos\theta = 0,\ \theta = 90^0$

Therefore the vectors are mutually perpendicular.

Example 1.2 A particle moves from a point (3,-4,-2) meter to another point (5,-6, 2) meter under the influence of a force $\vec{F} = (-3\hat{\imath} + 4\hat{J} + 4\hat{k})$ N. Calculate the work done by the force.

Solution: Suppose the particle moves from point A to B. Then displacement of particle is given by

$$\vec{r} = position\ vetor\ of\ B - position\ vetor\ A$$

$$\vec{r} = [(5-3)i + (-6+4)j + (2+2)k] \text{ meter}$$
$$\vec{r} = (2i - 2j + 4k) \text{ meter}$$

Work done = $\vec{F}.\vec{r}$ = $[(-3\hat{\imath} + 4\hat{\jmath} + 4\hat{k}).(2i - 2j + 4k)$ N-meter =2 joule.

1.3.2.2 Vector product or Cross Product

The vector product or cross product of two vectors is a vector quantity and defined as a vector whose magnitude is equal to the product of magnitudes of two vectors and sine of angle between them.

If $\vec{A}$ and $\vec{B}$ are two vectors then cross product of these two vectors is denoted by $\vec{A} \times \vec{B}$ (*read as* $\vec{A}\ cross\ \vec{B}$) and given as

$$\vec{A} \times \vec{B} = ABsin\emptyset\, \hat{n} = \vec{C}$$

Where $\emptyset$ is the angle between vectors $\vec{A}$ and $\vec{B}$, and $\hat{n}$ is the unit vector perpendicular to both $\vec{A}$ and $\overrightarrow{B}\left(i.e. normal\ to\ the\ plane\ containg\ \vec{A}\ and\ \vec{B}\right)$.

Suppose $\vec{A}$ is along x axis and $\vec{B}$ is along y axis then vector product can be considered as an area of parallelogram OPQR as shown is figure 1.11 in XY plane whose sides are $\vec{A}$ and $\vec{B}$ and direction is perpendicular to plane OPQR i.e. along z axis. The cross product $\vec{A}$ and $\vec{B}$ is positive if direction of $\emptyset$ ($\vec{A}\ to\ \vec{B}$) is positive or rotation is anticlockwise as show in figure 1.11, and negative if the rotation of $\emptyset$ ($\vec{A}\ to\ \vec{\boldsymbol{B}}$) is clockwise (figure 1.12).

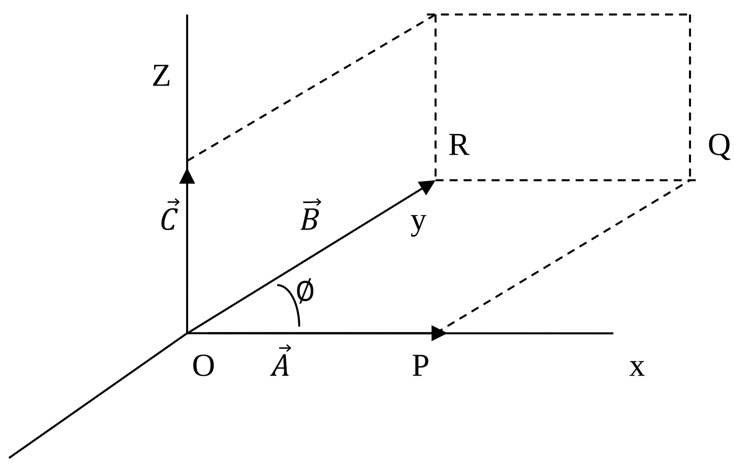


Figure1.11

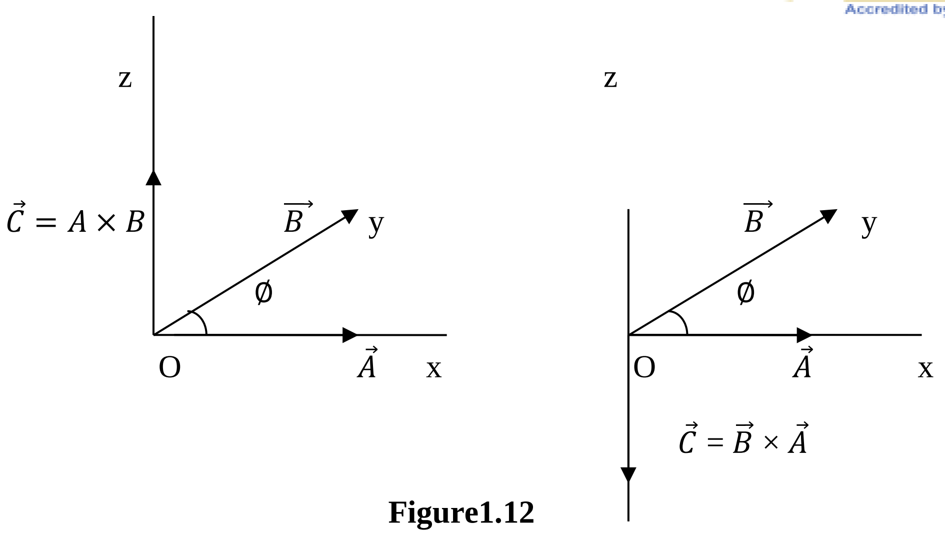


Figure1.12

Important properties of vector product

(i) Commutative law does not hold: From the definition of vector product of two vectors $\vec{A}$ and $\vec{B}$ the vector products are defined as

$$\vec{A} \times \vec{B} = \text{AB } \sin\emptyset\, \hat{n}$$
$$\vec{B} \times \vec{A} = \text{AB } \sin\emptyset\, (-\hat{n}) = -\text{AB } \sin\emptyset\, \hat{n} = -\, \vec{A} \times \vec{B}$$

Since in case of $\vec{B} \times \vec{A}$ the angle of rotation becomes opposite to case $\vec{A} \times \vec{B}$, hence product becomes negative.

Therefore, $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$

(ii) Distributive law holds:
In case of vector product the distribution law holds.

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

(iii) Product of equal vectors
If two vectors are equal then the angle between them is zero, and vector product becomes

$$\vec{A} \times \vec{A} = |A||A|sin\emptyset\, \hat{n} = 0$$

Hence the vector product of two equal vectors in always zero.

In case of Cartesian coordinate system if $\hat{\imath}, \hat{\jmath}, \hat{k}$ are unit vectors along x, y and z axes then

$$\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = 0$$

(iv) Collinear vectors: Collinear vectors are vectors parallel to each other. The angles between collinear vectors are always zero therefore

$$\vec{A} \times \vec{B} = |A||B|sin\emptyset\,\hat{n} = 0$$

Thus two vectors are parallel or anti-parallel or collinear if its vector product is 0.

(v) Vector product of orthogonal vector : If two vectors $\vec{A}$ and $\vec{B}$ are orthogonal to each other then angle between such vectors is $\emptyset = 90°$ therefore

$$\vec{A} \times \vec{B} = AB\ sin\emptyset\,\hat{n}$$
$$\vec{A} \times \vec{B} = |A||B|\,\hat{n}$$

In Cartesian coordinate system if $\hat{\imath}, \hat{\jmath}, \hat{k}$ are unit vector along x, y and z axes then

$$\boldsymbol{\hat{\imath} \times \hat{\jmath} = \hat{k} \quad \hat{\jmath} \times \hat{k} = \hat{\imath}\ and\ \hat{k} \times \hat{\imath} = \hat{\jmath}}$$
$$\hat{\jmath} \times \hat{\imath} = -\hat{k} \quad \hat{k} \times \hat{\jmath} = -\hat{\imath}\ and\ \hat{\imath} \times \hat{k} = \hat{\jmath}$$

(vi) Determinant form of vector product: If $\vec{A}$and $\vec{B}$ are two vectors given as

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$$
$$\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$$

Then

$$\vec{A} \times \vec{B} = \left(A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}\right) \times (B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k})$$
$$= A_xB_x\hat{\imath} \times \hat{\imath} + A_xB_y\hat{\imath} \times \hat{\jmath} + A_xB_z\hat{\imath} \times \hat{k} + A_yB_x\hat{\jmath} \times \hat{\imath} + A_yB_y\hat{\jmath} \times \hat{\jmath} +$$
$$+A_yB_z\hat{\jmath} \times \hat{k} + A_zB_x\hat{k} \times \hat{\imath} + A_zB_y\hat{k} \times \hat{\jmath} + A_zB_z\hat{k} \times \hat{k}$$

$$= A_xB_y\hat{k} - A_xB_z\hat{\jmath} - A_yB_x\hat{k} + A_yB_x\hat{\imath} + A_zB_x\hat{\jmath} - A_zB_y\hat{\imath}$$

(Since $\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = 0\ and\ \hat{\imath} \times \hat{k} = -\hat{\jmath}, \hat{\jmath} \times \hat{\imath} = -\hat{k}, \hat{k} \times \hat{\jmath} = -\hat{\imath}$)

$$= \hat{\imath}(A_yB_z - A_zB_y) - \hat{\jmath}\,(A_xB_z - A_zB_x) + \ k(A_xB_y - A_yB_x)$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Physical significance of vector product:

In physics, numbers of physical quantities are defined in terms of vector products. Some basic examples are illustrated below.

(i) **Torque:** Torque or moment of force is define as

$$\vec{\tau} = \vec{r} \times \vec{f}$$

Where $\vec{\tau}$ is torque, $\vec{r}$ is position vector of a point P where the force $\vec{f}$ is applied. (Figure 1.13)

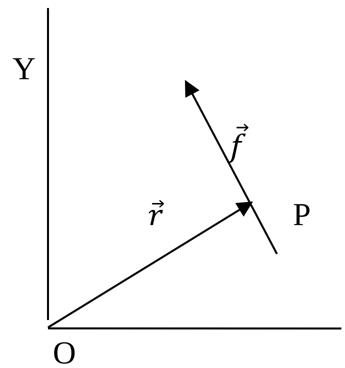


Figure 1.13

(ii) **Lorentz force on a moving charge in magnetic field**: if a charge q is moving in a magnetic field $\vec{B}$with a velocity $\vec{V}$at an angle with the direction of magnetic field then force $\vec{F}$experienced by the charged particle is give as;

$$\vec{F} = q(\vec{V} \times \vec{B})$$

This force is called Lorentz force and its direction is perpendicular to the direction of both velocity and magnetic field B.

(iii) **Angular Momentum**: Angular momentum is define as the moment of the momentum and given as:

$$\vec{L} = \vec{r} \times \vec{p}$$

Where $\vec{r}$ is the radial vector of circular motion and $\vec{p}$ is the linear moment of the body under circular motion, and $\vec{L}$ is angular momentum along the direction perpendicular to both $\vec{r}$ and $\vec{p}$. The law of conservation of angular momentum is a significant property in all circular motions.

1.3.3. Product of three vectors:

If we consider three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$, we can define two types of triple products known as scalar triple product and vector triple product.

1.3.3.1 Scalar Triple product:

Let us consider three vectors $\vec{A}$, $\vec{B}$and $\vec{C}$ then the scalar triple product of these three vectors is defined as $\vec{A}.(\vec{B}\times\vec{C})$ and denoted as $[\vec{A}\vec{B}\vec{C}]$. This is a scalar quantity. If we consider $\overrightarrow{A}$, $\vec{B}$ and $\vec{C}$ the three sides of a parallelopiped as shown in figure 1.14 then $\vec{B}\times\vec{C}$ is a vector which represents the area of parallelogram OBDC which is the base of the parallelogram. The direction of $\vec{B}\times\vec{C}$ is naturally along Z axis (perpendicular to both $\vec{B}$ and $\vec{C}$). If $\emptyset$ is the angle between the direction of vectors $(\vec{B}\times\vec{C})$ and vector $\vec{A}$, then the dot product of vectors $(\vec{B}\times\vec{C})$ and vector $\vec{A}$ is given as (figure 1.14)

$$\vec{A}\,.\,(\vec{B}\times\vec{C}) = |A||\vec{B}\times\vec{C}|cos\emptyset = ACos\emptyset(\vec{B}\times\vec{C}) = h.\,(\vec{B}\times\vec{C})$$

= Vertical height of parallelogram × area of base of parallelogram

= Volume of parallelogram $= [A\ B\ C]$.

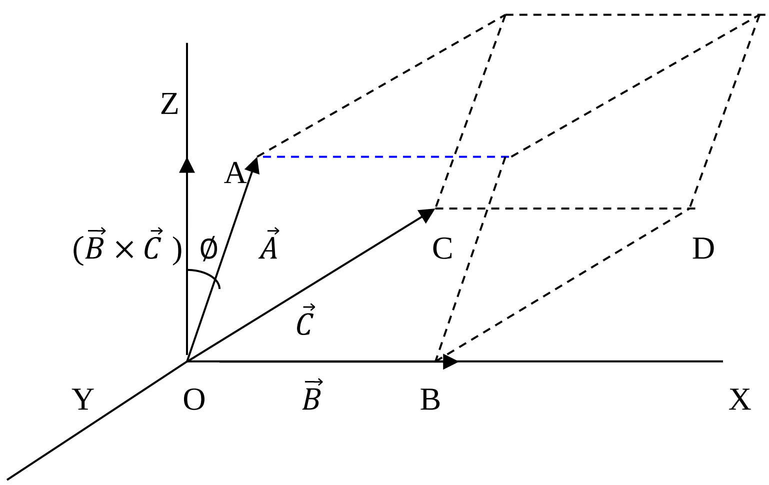


Figure 1.14

Therefore, it is clear that $\vec{A}\,.\,(\vec{B}\times\vec{C})$ represents the volume of parallelepiped constructed by vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ as its sides. Further, it is a scalar quantity as volume is scalar. It can also be noted that in case of scalar triple product the final product (volume of parallelepiped) remains same if the position of $\vec{A}, \vec{B}$ and $\vec{C}$ or dot and cross are interchanged.

$$[\vec{A}\,\vec{B}\,\vec{C}] = \vec{A}.(\vec{B}\times\vec{C}) = \vec{B}.(\vec{C}\times\vec{A}) = \vec{C}.(\vec{A}\times\vec{B}) = (\vec{B}\times\vec{C}).\vec{A} = (\vec{C}\times\vec{A}).\vec{B} = (\vec{A}\times\vec{B}).\vec{C}$$

Scalar triple product can also be explained by determinant as

$$[\vec{A}\,\vec{B}\,\vec{C}] = \vec{A}.(\vec{B}\times\vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

In case of three vectors to be coplanar, it is not possible to construct a parallelepiped by using such three vectors as its sides; therefore the scalar triple product must be zero.

$$[\vec{A}\,\vec{B}\,\vec{C}] = \vec{A}.(\vec{B}\times\vec{C}) = 0$$

1.3.3.2 Vector triple product:

The vector triple product of three vectors is define as

$$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}.\vec{C})\vec{B} - (\vec{A}.\vec{B})\vec{C}$$

The vector triple product is product of a vector with the product of two another vectors. The vector triple product can be evaluated by determinant method as given below.

$$(\vec{B}\times\vec{C}) = \begin{vmatrix} i & j & k \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$= i(B_yC_z - B_zC_y) - j(B_xC_z - B_zC_x) + k(B_xC_y - B_yC_x)$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_yC_z - B_zC_y & B_zC_x - B_xC_z & B_xC_y - B_yC_x \end{vmatrix}$$

$$= (\vec{A}.\vec{C})\vec{B} - (\vec{A}.\vec{B})\vec{C}$$

As in cross product the vector $\vec{A}\times(\vec{B}\times\vec{C})$ will be perpendicular to plane containing vectors $\vec{A}$ and $(\vec{B}\times\vec{C})$. Since $(\vec{B}\times\vec{C})$ is itself in the direction perpendicular to plane containing $\vec{B}$ and $\vec{C}$, therefore the direction of $\vec{A}\times(\vec{B}\times\vec{C})$ will be along the plan containing $\vec{B}$ and $\vec{C}$, $hence\ is\ a\ linear$ combination of $\overrightarrow{B}\ and\ \overrightarrow{C}$.

1.4 Summary

1. Physical quantities are of two types, scalar and vector. The scalar quantities have magnitude only but no direction. The vector quantities have magnitude as well as direction.
2. Two vector quantities can be added with parallelogram law and triangle law. In parallelogram law, the resultant is denoted by the diagonal of parallelogram whose adjacent sides are represented by two vectors. In triangle law, we place the tail of second vector on the head of first vector, and resultant is obtained by a vector whose head is at the head of second vector and tail is at the tail of first vector.
3. For subtraction, we reverse the direction of second vector and add it with first vector.
4. In case of more than two vectors we simply use Polygon law of vector addition.
5. Any vector can be resolved into two or more components. By adding all components we can find the final vector.
6. If a vector makes angles α, β and γ with three mutual perpendicular axes x, y and z respectively then $cos\ \alpha, cos\ \beta$ and $cos\ \gamma$ are called direction cosines.
7. Scalar product of two vectors is defined as $\overrightarrow{P}.\ \vec{Q}\ = PQ\cos\theta$ which is a scalar quantity.
8. Vector product of two vectors is defined as $\vec{A}\times\vec{B} = ABsin\emptyset\ \hat{n}$ which is a vector quantity. The direction of vector is perpendicular to $\vec{A}$ and $\vec{B}$.
9. If two vectors are parallel to each other then they are said to be collinear. For collinear vectors $\overrightarrow{P}.\ \vec{Q}\ = PQ$ or $\vec{P}\times\vec{Q} = 0$
10. If the angle between two vectors is 90^0, then vectors are called orthogonal. In this case $\overrightarrow{P}.\ \vec{Q}\ = 0$
11. Cross product of two vectors can also be calculated by determinant. The determinant form of cross product is
$$\vec{A}\times\vec{B} = \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$
12. Scalar triple product of three vectors can also be calculated by determinant. The determinant form of Scalar triple product is
$$\overrightarrow{A}.(\vec{B}\times\vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$
13. Vector triple product is defined as
$$\vec{A}\times\left(\vec{B}\times\vec{C}\right) = \left(\vec{A}.\vec{C}\right)\vec{B} - (\vec{A}.\vec{B})\vec{C}$$

1.5 Glossary

Vector- Physical quantity with direction

Scalar quantities- Physical quantity without direction
Collinear – in same line or direction
Orthogonal- perpendicular to each other
Coplanar – on same plane

1.6 Reference Books

1. Mechanics – D.S. Mathur, S Chand, Delhi
2. Concept of Physics- H C Verma, Bharti Bhawan, Patna
3. Physics Part-I, Robert Resnick and David Halliday, Wiley Eastern Ltd

1.7 Suggested readings

1. Modern Physics, Beiser, Tata McGraw Hill
2. Fundamental University Physics-I, M. Alonslo and E Finn, Addition-Wesley Publication
3. Berkeley Physics Course Vol I, Mechanics, C Kittel et al, McGraw- Hill Company

1.8 Terminal questions

1.8.1 Short answer type questions

1. Define unit vector, like vector and equal vectors.
2. What are direction cosines? Give its significance.
3. What angle does the vector $3i + \sqrt{2}j + k$ make with y axis?
4. What is the condition for vector to be collinear?
5. Explain the difference between dot and cross products.
6. What is angular momentum? How the direction of angular momentum can be decided?
7. Give some examples of dot product in physics.
8. Give some examples of cross product in physics.
9. Define scalar triple product.
10. How the angle between two vectors can be obtained?

1.8.2 Essay type questions

1. If $|\boldsymbol{A} + \boldsymbol{B}| = |\boldsymbol{A} - \boldsymbol{B}|$, show that **A** and **B** are perpendicular to each other.
2. What is the significance of dot product? Give the properties of cross product.
3. Show that $A = 5i + 2j + 4k \; and \; B = 2i + 3j - 4k$ are perpendicular to each other.
4. What is the vector product? Give the properties of vector product.
5. Find out the condition if two vectors are collinear.
6. Find the components of a vector along and perpendicular to the direction of another vector.

1.8.3 Numerical question

1. Calculate the dot product of vectors $\boldsymbol{A} = 6i + 7j + k \; and \; \boldsymbol{B} = i + 3j + 2k$.
2. A particle moves from the position $(3i + 3j + 2k) meter$ to another position (-$2i + 2j + 4k$) $meter$ under the influence of a force $\boldsymbol{F} = 3i + 2j + 4k$ newton. Calculate the work done by the force.
3. Obtain the projection of a vector $\boldsymbol{A} = 3i + 4j + 5k$ along a line which originates at a point (2, 2, 0) and passing through another point (-2, 4, 4).
4. Find the unit vector in the direction of resultant vectors of $\boldsymbol{A} = 6i + 7j + k \; and \; \boldsymbol{B} = i + 3j + 2k$.

UNIT 2: VECTOR CALCULUS

STRUCTURE:

2.0 Objective:

In previous unit we studied about the basic concepts of vector like its meaning, significance, representation, addition, subtraction etc. Now in this unit, we will learn some further use of vectors in physics and mathematics. After reading this unit we will able to understand:

1. Differentiation of vector
2. Del operator
3. Scalar and vector fields
4. Gradient
5. Curl
6. Divergence
7. Vector identities
8. Applications in physics

2.1 Introduction:

Differentiation and integration techniques are frequently used in physics and mathematics. Therefore this unit is basically vector calculus. Theses calculus techniques are used to solve and explain many physical problems. In this unit we will understand the differentiation and integration of vector quantities. Further we define some new terms like gradient, curl, divergence, its properties and application. The physical significance of these terms will also be discussed in detail.

2.2 Differentiation of vector:

Suppose $\vec{r}$ is the position vector of a particle situated at point P with respect to origin O. If particle moves with time, then vector $\vec{r}$ varies corresponding to time t, and $\vec{r}$ is said to be vector function of scalar variable t and represented as $\vec{r}$ =F(t)

If P is the position of particle at time t then OP = $\vec{r}$

If Q is the position of particle at time t+δt and position vector of Q is $(\vec{r} + \delta\vec{r})$

then $\overrightarrow{PQ} = \overrightarrow{OQ}$-$\overrightarrow{OP}$

$= \vec{r} + \delta\vec{r} - \vec{r}$

In limiting case if $\delta t \to 0$ then $\delta\vec{r} \to 0$ and P tends to Q and the chord become the tangent at P. Differentiation is define as

$$\frac{d\vec{r}}{dt} = \lim_{\delta t \to 0} \frac{\delta\vec{r}}{\delta t} = \lim_{\delta t \to 0} \frac{\vec{r}(t+\delta t) - \vec{r}(t)}{\delta t}$$

When the limit exists only then the function $\vec{r}$ is differentiable. If we further differentiate function with respect t then it is called second order differentiation. If should be cleared that the derivatives of a vector (say $\vec{r}$) are also vector quantities.

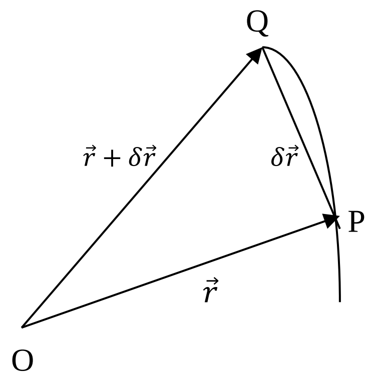


Figure 2.1

2.2.1 Properties of vector differentiation:

If $\vec{A}$ and $\vec{B}$ are two vectors, $\emptyset$ is a scalar field and $\vec{C}$ is a constant vector then

(1) $\frac{d}{dt}(\vec{A}+\vec{B})=\frac{d\vec{A}}{dt}+\frac{d\vec{B}}{dt}$

(2) $\frac{d}{dt}(A\times\emptyset)=\frac{d\vec{A}}{dt}\emptyset+\vec{A}\frac{d\emptyset}{dt}$

(3) $\frac{d}{dt}(\vec{A}.\vec{B})=\vec{A}.\frac{d\vec{B}}{dt}+\frac{d\vec{A}}{dt}.\vec{B}$

(4) $\frac{d}{dt}(\vec{A}\times\vec{B})=\vec{A}\times\frac{d\vec{B}}{dt}+\frac{d\vec{A}}{dt}\times\vec{B}$

(5) $\frac{d\vec{C}}{dt}=0$

(6) $\frac{d\vec{r}}{dt}=\frac{d\vec{r}}{ds}\frac{ds}{dt}$ if s is scalar function of t.

(7) $\frac{d}{dt}(r^2)=\frac{d}{dt}(\vec{r}.\vec{r})=\vec{r}\frac{d\vec{r}}{dt}+\vec{r}\frac{d\vec{r}}{dt}=2\vec{r}\frac{d\vec{r}}{dt}$, if $\vec{r}$ is position vector.

Example 2.1: A particle is moving along the curve x $=t^2+2$, y= t^2+1 and z= $3t+5$. Find the velocity and acceleration of particle along the direction 3i+2j+6k at time t=2.
Solution:
Curve is define as x $=t^2+2$, y= t^2+1 and z= $3t+5$.
The position vector of particle at any time t is given as

$$\bar{r}=xi+yj+zk$$
$$\bar{r}=(t^2+2\,)i+(t^2+1)j+(3t+5)k$$

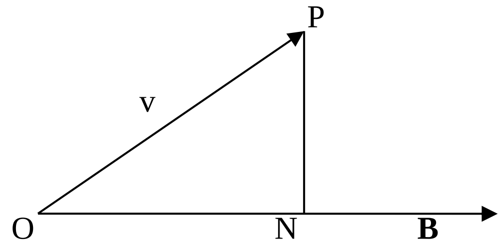


Figure 2.2

Velocity is given as

$$\frac{d\bar{r}}{dt} = 3t^2 i + 2tj + 3k$$

at t=2 velocity becomes

$$\frac{d\bar{r}}{dt} = 12i + 4j + 3k$$

Component of the velocity along the direction $3i + 2j + 6k = \vec{B}\ (say\ )$

$$ON = |\bar{v}| \cos\theta \,.\hat{b} = |\bar{v}|\ \frac{\bar{v}.\bar{B}}{|\bar{v}||\bar{B}|}.\frac{B}{|\bar{B}|} = \frac{(\bar{v}.\bar{B})B}{|B|^2}$$

$$= \frac{(16\ i+4j+3k).(3i+2j+6k)}{3^2+2^2+6^2}(3i + 2j + 6k) = \frac{74}{49}(3i + 2j + 6k)$$

acceleration $\bar{a}$ can be given as $\bar{a} = \frac{d\bar{r}}{dt} = 6t\ i + 2j$

acceleration $\bar{a}$ at t=2 can be given as $\bar{a} = 12i + 2j$

Component of acceleration along direction $\bar{B}$ is given as

$$= |\bar{a}| \cos\theta \,.\ \hat{b}\ = |\bar{a}|\frac{\bar{a}.\bar{B}}{|\bar{a}|\,|B|}\frac{\bar{B}}{|B|} = \frac{(\bar{a}.\bar{B})\,\bar{B}}{|B|^2}$$

$$= \frac{(12i+2j).(3i+2j+6k)}{32^2+2^2+6^2}(3i + 2j + 6k)$$

$$= \frac{52}{49}(3i + 2j + 6k)$$

2.2.2 Partial derivative:

If f is a vector function which depends on variable (x, y, z), then the partial derivatives are defined as

$$\frac{\partial f}{\partial x} = \lim_{\delta x \to 0} \frac{f(x + \delta x, y, z) - f(x, y, z)}{\delta x}$$

$$\frac{\partial f}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y + \delta y, z) - f(x, y, z)}{\delta y}$$

$$\frac{\partial f}{\partial z} = \lim_{\delta z \to 0} \frac{f(x, y, z + \delta z) - f(x, y, z)}{\delta z}$$

In case of partial derivatives with respect to a variable, all the other remaining variables are taken as constant.

Partial derivatives of second order are defined as:

$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} (\frac{\partial f}{\partial x})$

$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} (\frac{\partial f}{\partial y})$

$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z} (\frac{\partial f}{\partial z})$

2.2.3 Del operator:

The vector differential operator del is denoted by ∇ and is defined as

$$\nabla = \mathrm{i}\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

2.2.4 Scalar and vector point functions:

(1) Field: Field is a region of the space defined by a function.

(ii) Scalar point function: A scalar function $\emptyset(x, y, z)$ defines all scalar point in the space. For example, gravitational potential is a scalar function defined at all gravitational fields in the space.

(iii) Vector potential function: If a vector function $\vec{F}$(x, y, z) defines a vector at every point in space then it is called vector point function. For example gravitational force is a vector function defined at a gravitational field in the space.

2.2.5 Gradient:

The gradient of a scalar function $\emptyset$ is defined as

grad $\emptyset = \nabla\emptyset = (\mathrm{i}\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z})\emptyset$

$= \mathrm{i}\frac{\partial \emptyset}{\partial x} + j\frac{\partial \emptyset}{\partial y} + k\frac{\partial \emptyset}{\partial z}$

grad $\emptyset$ *is a vector qunatity.*

Total differential d$\emptyset$ of a scalar function $\emptyset(x, y, z)$ can be expressed as,

$$d\emptyset = \frac{\partial\emptyset}{\partial x}dx + \frac{\partial\emptyset}{\partial y}dy + \frac{\partial\emptyset}{\partial z}dz$$

Total differential d∅ of a scalar function ∅ can also be expressed as

$d\emptyset = \frac{\partial\emptyset}{\partial x}dx + \frac{\partial\emptyset}{\partial y}dy + \frac{\partial\emptyset}{\partial z}dz$

$= (i\frac{\partial\emptyset}{\partial x} + j\frac{\partial\emptyset}{\partial y} + k\frac{\partial\emptyset}{\partial z})\ (idx + jdy + kdz)$

$d\emptyset = (\vec{\nabla}\emptyset).\vec{dr} = |\nabla\emptyset||dr|\cos\theta = (\vec{\nabla}\emptyset).\ dr\ \hat{r}$, (where $\hat{r}$ is a unit vector along $d\vec{r}$)

also θ is angle between $\vec{\nabla}\emptyset$ and $d\vec{r}$ (The direction of displacement).

So, $\frac{d\emptyset}{dr} = (\vec{\nabla}\emptyset).\ \hat{r}$

Thus , $\frac{d\emptyset}{dr}$ is the directional derevative of ∅. The rate of change is maximum if $\hat{r}$ is along $\vec{\nabla}\emptyset$ i.e. angle between $\vec{\nabla}\emptyset$ and $\hat{r}$ is zero.

Hence gradient of the scalar field ∅ defines a vector field, the magnitude of which is equal to the maximum rate of change of ∅ and the direction of which is the same, as the direction of displacement along with the rate of change is maximum.

Example 2.2:
In the heat transfer, the temperature of any point in space is given by $= xy + yz + zx$. Find the gradient of T in the direction of vector 4i-3k at a point (2, 2, 2).

Solution:

Temperature is define as

$T = xy + yz + zx$

gradient of temperature T is given as

$$grad\ T = \nabla T = \left(i\frac{\partial}{\partial x} + j\ \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\partial\right)(xy + yz + zx)$$

$$\nabla T = i(y + z) + j(x + z) + k(x + y)$$

at point (2, 2, 2) the ∇T is $(4i + 4j + 4k)$

The gradient T in the direction of vector 4i-3k is

$= (4i + 4j + 4k)$. Unit vector along $(4i - 3k)$

$=(4i + 4j + 4k).\frac{(4i-3k)}{\sqrt{4^2+3^2}}$

=4/5

2.26 Physical significance of grad ∅ :

The physical significance of grad ∅ can be explained on the basis of surface defined by scalar field ∅. The value of ∅ remains constant on the surface S, as shown in figure 2.3 and it is called a level surface or equi-scalar surface. Let us consider two surfaces S and S' defined by scalar function ∅ and ∅+d∅ respectively. Suppose $\vec{n}$ is normal to the surfaces S and S'. If the coordinates of point P and Q are (x, y, z) and (x+dx, y+dy, z+dz) then the distance between P and Q are

$$d\vec{r} = idx + jdy + kdz$$

as the definition of differentiation

$$d\emptyset = \frac{\partial\emptyset}{\partial x}dx + \frac{\partial\emptyset}{\partial y}dy + \frac{\partial\emptyset}{\partial z}dz$$

$$= \left(\frac{\partial\emptyset}{\partial x}i + \frac{\partial\emptyset}{\partial y}y + \frac{\partial\emptyset}{\partial z}k\right).(dx\ i + dy\ j + dz\ k)$$

$$d\emptyset = \vec{\nabla}\emptyset.d\vec{r}$$

If we consider the point Q approaches to P and finally lies on P then

$$d\emptyset = 0$$

$$\vec{\nabla}\emptyset.d\vec{r} = 0$$

$\nabla\emptyset\ and\ dr\ are\ perpendiular\ to\ each\ other.$

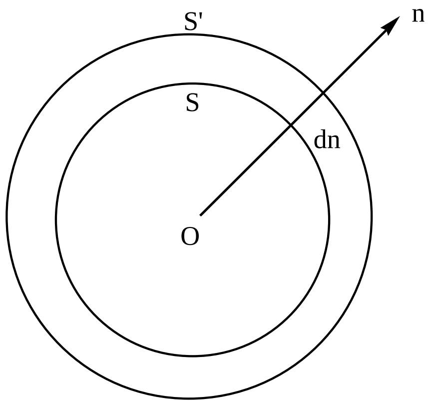


Figure 2.3

Therefore, $\nabla\emptyset$ is a vector which is perpendicular to the surface S.

If $\vec{n}$ is normal on the surface S and d$\vec{n}$ represents the distance between surfaces S to S' then $dn = dr\cos\theta = \hat{n}.d\vec{r}$

And $d\emptyset = \frac{\partial\emptyset}{\partial n}dn = \frac{\partial\emptyset}{\partial n}\hat{n}.d\vec{r}$

By using equation (1), $\vec{\nabla}\emptyset.d\vec{r} = \frac{\partial\emptyset}{\partial n}\hat{n}.d\vec{r}$

$$\vec{\nabla}\emptyset = \frac{\partial\emptyset}{\partial n}\hat{n}$$

Thus, $\nabla\emptyset$ is defined as a vector whose magnitude is rate of change of $\emptyset$ along normal to the surface and direction is along the normal to the surface.

Example2.3:

Find the directional derivative of a scalar function $\emptyset(x, y, z) = x^2 + xy + z^2$ at the point A (2,-1,-1) in the direction of the line AB where coordinate of B are (3, 2, 1).

Solution:

The component of $\nabla\emptyset$ along the direction of a vector $\vec{A}$ is called directional derivative of $\emptyset$ and given as $\nabla\emptyset . \hat{A}$

Now $\nabla\emptyset = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2 + xy + z^2)$

$= (2x + y)i + x\,j + 2z\,k$

gradient at point A (2,-1,-1)

$\nabla\emptyset = 3i + 2\,j - 2\,k$

The vector $\overrightarrow{AB} = position\ vector\ of\ B - position\ vector\ of\ A$

$= (3i + 2j + k) - (2i - j - k) = i + 3j$

Directional derivative of $\emptyset$ in the direction of AB is

$$\vec{\nabla}\emptyset\,.\,\widehat{AB} = (3i + 2j - 2k).\frac{(i + 3j)}{\sqrt{1+9}} = \frac{9}{\sqrt{10}}$$

2.3 Divergence of Vector:

The divergence is defined as dot product of del operator with any vector point function $\vec{f}$ or any vector $\bar{F}$ and given as,

div. $\vec{f} = \nabla.\vec{f} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right).\,(if_x + jf_{y+}kf_z)$ where $\vec{f} = if_x + jf_{y+}kf_z$

$$= \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}$$

Since divergence of a vector $\vec{f}$ is dot product of del operator $\vec{\nabla}$ and that vector $\vec{f}$, therefore it is a scalar quantity.

2.3.1 Physical Significance of Divergence:

On the basis of fluid dynamics or a fluid flow, the divergence of a vector quantity can be explained. Let us consider a parallelepiped of edges dx, dy and dz along the x, y, z directions as shown in figure 2.4.

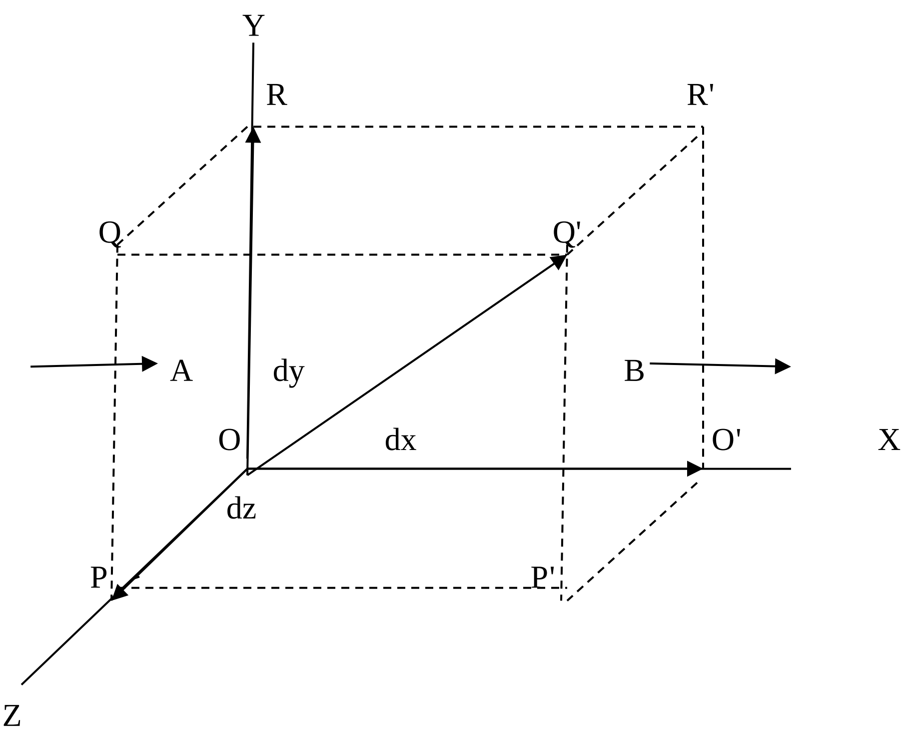


Figure 2.4

Let $\vec{v}$ is the velocity of fluid at A(x, y, z) and given as

$$\vec{v} = v_x i + v_y j + v_z k$$

Where the $v_{x,}v_y, v_z$ *are the components of veolcity along x, y, z directions.*

Amount of fluid entering through the surface O'P'Q'R' per unit time is given as:

$$velocity \times area = v_x dydz$$

Amount of fluid flowing out through the surface O'P'Q'R' per unit times is given as

= $v_{x+dx}\, dydz$

=$(v_x + \frac{\partial v_x}{\partial x} dx)dydz$

Decrease in the amount of fluid in the parallelepiped along x axis per unit time.

=$v_x dydz - \left(v_x + \frac{\partial v_x}{\partial x} dx\right) dydz$

= $-\frac{\partial v_x}{\partial x} dxdydz$

Negative sign shows, decrease in the amount of fluid inside the parallelepiped.

Similarly decrease of amount of fluid along y axis

=$-\frac{\partial v_y}{\partial y} dxdydz$

Decrease of amount of fluid along z axis

= $-\frac{\partial v_z}{\partial z} dxdydz$

Total amount of fluid decrease inside the parallelepiped per unit time= $-\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) dxdydz$)

Thus the rate of loss of fluid per unit volume $= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

(We can ignore negative sign when we specify that the negative sign indicates decrease in the amount of fluid).

Further the rate of loss of fluid per unit volume

=$\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(v_x i + v_y j + v_z k)$ = $\vec{\nabla}.\vec{v} = div\ \vec{v}$

Thus, *the divergence of velocity vector shows the rate of loss of fluid per unit timer per unit volume.*

If we consider fluid is incompressible, there is not any loss or gain in the amount of fluid, therefore $div\ v = 0$

If the divergence of a vector is 0, then the vector function is called solenoidal.

Example 2.4: if $u=x^2+y^2+z^2$ and $\bar{r} = 2xi + 3yj + 2zk$, then find the div ($u\bar{r}$).

Solution : Div ($u\,\bar{r}$) $= \nabla.(u\,\bar{r})$

$$\left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right).\left[(x^2+y^2+z^2)(2xi+3yj+2zk)\right]$$

$$= i\frac{\partial}{\partial x}(x^2 2x)i + j\frac{\partial}{\partial y}(y^2 3y)j + k\frac{\partial}{\partial z}(z^2.2z)k$$

$$= 6x^2+9y^2+6z^2$$

2.4 Curl

The curl of a vector $\vec{F}$ is defined as

Curl $\overline{F} = \nabla \times \overline{F}$ (where $\bar{F}=F_x i + F_y j + F_j k$)

$=(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}) \times (F_x i + F_y j + F_j k)$

In terms of determinant of vector product

Curl $\bar{F}=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$

Since curl is vector product of two vectors, therefore it is a vector quantity.

2.4.1 Physical significance of curl:

On the basis of angular velocity and linear velocity the curl can be explained.

Let us consider a particle moving with velocity $\bar{v}$ and $\bar{r}$ is the position vector of particle rotating around origin O. Let $\vec{\omega}$ is the angular velocity of particle then

$$curl\,\bar{v} = \nabla \times \bar{v}$$

$$= \nabla \times (\bar{\omega} \times \bar{r}) \qquad (\because\ \bar{v} = \bar{\omega} \times \bar{r})$$

$$= \nabla(\omega_x i + \omega_y j + \omega_z k) \times (xi+yj+zk)$$

$$= \nabla \times \begin{vmatrix} i & j & k \\ \omega_x & \omega_y & \omega_z \\ x & y & z \end{vmatrix}$$

$$= \nabla \times \left[\left(\omega_y z - \omega_z y \right) i - \left(\omega_x z - \omega_z x \right) j + \left(\omega_x y - \omega_y x \right) k \right]$$

$$= \begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_y z - \omega_z y & \omega_z x - \omega_x z & \omega_x y - \omega_y x \end{bmatrix}$$

$$curl\,\overline{v} = 2\left(\omega_x i + \omega_y j + \omega_z k \right) = 2\overline{\omega}$$

Thus the curl of velocity shows angular velocity which means rotation of particle. Thus curl of a vector quantity is connected with rotational properties of vector field. If curl of a vector is zero, $\nabla \times \overline{f} = 0$ then there is no rotational property and $\overline{f}$ is called irrotational.

Example 2.5

Calculate the curl of a vector given by $\overline{F} = xyzi + 2x^2yj + (x^2z^2 - 2y^2)$k.

Solution:

$$curl\,\overline{F} = \nabla \times \overline{F}$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) \times (xyzi + 2x^2yj + (x^2z^2 - 2y^2)k)$$

$$= \begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & 2x^2y & x^2z^2 - 2y^2 \end{bmatrix}$$

$$= -4yi - (2xz^2 - xy)j + (4xy - xz)k$$

Example2.6:

Show that $F = (y^2 + 2xz^2)i + (2xy - z)j + (2x^2z - y + 2z)\vec{k}\ is\ irrolational.$

Solution:

$$curl\,F = \nabla \times F$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) \times [(y^2 + 2xz^2)i + (2xy - z)j + (2x^2z - y + 2z)\vec{k}]$$

$$= 0$$

Therefore F is irrotational.

2.5 Vector integral:

2.5.1 Line Integral: The integral of a vector function $\vec{F}$ along a line or curve is called line integral.

Suppose $\vec{F}$(x, y, z) be a vector function and PQ is a curve and $\overrightarrow{dl}$ is a small length of curve then line integral of vector $\vec{F}$ along a length $\overrightarrow{dl}$ is given as

$$\int_l \vec{F}.d\vec{l}$$

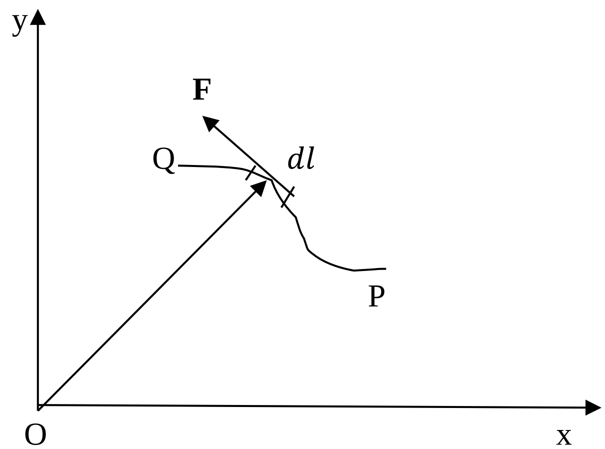


Figure 2.5

The integral may be closed or open depending on the nature of the curve whether closed or open. To compute the line integral of a function F, any method of integral calculus may be employed. In case of fore $\vec{F}$ acting on a particle along a curve PQ, the total work done can be calculated as line integral of force.

Work done= $\int_p^Q \vec{F}.\overrightarrow{dl}$

2.5.2 Surface integral:

Similarly as line integral of F is a vector function and s is a surface, then surface integral of a vector function F over the surface s is given as

Surface integral=$\iint_s \vec{F}.\overrightarrow{dl}$

The direction of surface integral is taken as perpendicular to the surface s.

If ds is written as ds=dxdy

Surface integral=$\iint_s \vec{F}.d\vec{s} = \int_x \int_y F.dxdy$

Surface integral represents flux through the surface S.

2.5.3 Volume integral:

If dV denotes the volume defined by $dxdydz$ then the volume integration of a vector F is define as

Volume integral= $\int_V FdV = \int_x \int_y \int_z F.dxdydz$

The volume integral can be explained in terms of total charge inside a volume. Suppose ρ is charge density of a volume dV then total charge inside the volume is given as q= $\int_v \rho \, dV$

2.6 Vector identities:

If $\emptyset_1$ and $\emptyset_2$ are two scalar point functions and $\vec{A}$ and $\vec{B}$ are two vectors, then

$$\nabla(\emptyset_1 + \emptyset_2) = \nabla\emptyset_1 + \nabla\emptyset_2$$

$$\nabla(\emptyset_1\emptyset_2) = \emptyset_1\nabla\emptyset_2 + \emptyset_2\nabla\emptyset_1$$

$$div\,(\vec{A} + \vec{B}) = div\,\vec{A} + div\,\vec{B}$$

$$div\,(\vec{A}.\vec{B}) = \overrightarrow{A}.\,div\,B + \overrightarrow{B}.\,div\,\vec{A}$$

$$curl\,(\vec{A} + \vec{B}) = curl\,\vec{A} + curl\,\vec{B}$$

$$div\,(\emptyset\,\vec{A}) = \emptyset\,div\,\vec{A} + \vec{A}\,.grad\,\emptyset$$

$$curl\,(\emptyset\vec{A}) = \emptyset\,curl\,\vec{A} + grad\,\emptyset\,\times\vec{A}$$

$$div\,curl\,\vec{A} = 0$$

$$curl\,grad\,\,\emptyset = 0$$

$$div\,(\vec{A} \times \vec{B}) = \vec{B}.curl\,\vec{A} + \vec{A}.curl\,\vec{B}$$

$$curl\,curl\,\vec{A} = grad\,div\,\vec{A} - \nabla^2\,\vec{A}$$

Example2.7 : Prove that

(1) $div\,curl\,\vec{A} = 0$
(2) $curl\,grad\,\,\emptyset = 0$

Solution:

(1) (1) $div\,curl\,\vec{A} = \nabla.\nabla \times \vec{A}$

$$= \nabla . \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$=\nabla . \left[i \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + j (\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}) + \mathrm{k} (\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}) \right]$$

$$= \frac{\partial}{\partial x} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \frac{\partial}{\partial y} (\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}) + \frac{\partial}{\partial z} (\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y})$$

$$=0$$

(2) $curl\ grad\ \emptyset = \nabla \times \nabla \emptyset$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \emptyset}{\partial x} & \frac{\partial \emptyset}{\partial y} & \frac{\partial \emptyset}{\partial z} \end{vmatrix}$$

$$= i \left(\frac{\partial^2 \emptyset}{\partial y \partial z} - \frac{\partial^2 \emptyset}{\partial z \partial y} \right) + j \left(\frac{\partial^2 \emptyset}{\partial z \partial x} - \frac{\partial^2 \emptyset}{\partial x \partial z} \right) + k \left(\frac{\partial^2 \emptyset}{\partial x \partial y} - \frac{\partial^2 \emptyset}{\partial y \partial x} \right) = 0$$

Example2.8:

Show that

(i) $div\left(\vec{A} \times \vec{B}\right) = \vec{B}. curl\ \vec{A} - \vec{A}\,. curl\ \vec{B}$

(ii) $curl\ curl\ \vec{A} = grad\ div\ \vec{A} - \nabla^2\ \vec{A}$

Solution (i) $div\left(\vec{A} \times \vec{B}\right) = \nabla . (\vec{A} \times \vec{B})$

$$= (\hat{\imath}\ \frac{\partial}{\partial x} + \hat{\jmath}\ \frac{\partial}{\partial y} + \hat{k}\ \frac{\partial}{\partial z}). [(A_yB_z - A_zB_y)\ \hat{\imath} + (A_zB_x - A_xB_z)\ \hat{\jmath} + (A_x B_y - A_yB_x)\hat{k}]$$

$$= \frac{\partial}{\partial x} (A_yB_z - A_zB_y) + \frac{\partial}{\partial y} (A_zB_x - A_xB_z) + \frac{\partial}{\partial z} (A_xB_y - A_yB_x)$$

$$= B_x(\frac{\partial Az}{\partial y} - \frac{\partial Ay}{\partial z}) + B_y(\frac{\partial Ax}{\partial z} - \frac{\partial Az}{\partial x}) + B_z(\frac{\partial Ay}{\partial x} - \frac{\partial Ax}{\partial y}) - A_x(\frac{\partial Bz}{\partial y} - \frac{\partial By}{\partial z}) - A_y(\frac{\partial Bx}{\partial z} - \frac{\partial Bz}{\partial x})$$

$$- A_z(\frac{\partial By}{\partial x} - \frac{\partial Bx}{\partial y})$$

$$= (B_x\hat{\imath} + B_y\hat{\jmath} + Bz\hat{k}). [(\frac{\partial Az}{\partial y} - \frac{\partial Ay}{\partial z})\hat{\imath} + (\frac{\partial Ax}{\partial z} - \frac{\partial Az}{\partial x})\hat{\jmath} + (\frac{\partial Ay}{\partial x} - \frac{\partial Ax}{\partial y})\hat{k}] -$$

$$(A_x\hat{\imath} + A_y\hat{\jmath} + Az\hat{k}). [(\frac{\partial Bz}{\partial y} - \frac{\partial By}{\partial z})\hat{\imath} + (\frac{\partial Bx}{\partial z} - \frac{\partial Bz}{\partial x})\hat{\jmath} + (\frac{\partial By}{\partial x} - \frac{\partial Bx}{\partial y})\hat{k}]$$

$$= \overrightarrow{B}.\ \mathrm{curl}\ \vec{A} - \overrightarrow{A}.\ \mathrm{curl}\ \vec{B}$$

$$= \mathrm{curl}\ \vec{A}\ .\ \vec{B} - \mathrm{curl}\ \vec{B}\ .\ \vec{A}$$

Solution (ii)

$$curl\ curl\ \bar{A} = \nabla \times (\nabla \times \bar{A})$$

$$= (i\frac{\partial}{\partial x} + j\ \frac{\partial}{\partial y} + k\frac{\partial}{\partial z}) \times \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= (i\frac{\partial}{\partial x} + j\ \frac{\partial}{\partial y} + k\frac{\partial}{\partial z}) \times \left[i\ \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - j(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}) + \mathrm{k}(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y})\right]$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \left(\frac{\partial A_y}{\partial y} - \frac{\partial A_y}{\partial z}\right) & (\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}) & (\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}) \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}\ \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - \frac{\partial}{\partial z}(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x})\right] + j\left[\frac{\partial}{\partial z}\ \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \frac{\partial}{\partial x}(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y})\right]$$
$$+k\left[\frac{\partial}{\partial x}\ \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) - \frac{\partial}{\partial y}(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z})\right]$$

$$= i\left[\frac{\partial^2 A_y}{\partial y \partial x} - \frac{\partial^2 A_x}{\partial y^2} - \frac{\partial^2 A_x}{\partial z^2} + \frac{\partial^2 A_z}{\partial z \partial x}\right] + j\left[\frac{\partial^2 A_z}{\partial z \partial y} - \frac{\partial^2 A_y}{\partial z^2} - \frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_x}{\partial x \partial y}\right]$$
$$+k\left[\frac{\partial^2 A_x}{\partial x \partial z} - \frac{\partial^2 A_z}{\partial x^2} - \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_y}{\partial y \partial z}\right]$$

$$= \sum i\left[\left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial y \partial x} + \frac{\partial^2 A_z}{\partial z \partial x}\right) - \ (\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2})\right]$$

$$= \sum i\frac{\partial}{\partial x}\ \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) - \sum i\left[\left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}\right)\right]$$

$$= grad\ div\ \bar{A} - \nabla^2\ \bar{A}$$

2.7 Summary:

1. Differentiation and integration techniques are used to solve and explain many physical problems. Differentiation of a vector is defined as

$$\frac{d\vec{r}}{dt} = \lim_{\delta t \to 0} \frac{\delta\vec{r}}{\delta t} = \lim_{\delta t \to 0} \frac{\vec{r}(t+\delta t) - \vec{r}(t)}{\delta t}$$

2. If we further differentiate function with respect t then it is called second order differentiation. If should be cleared that the derivatives of a vector (say $\vec{r}$) are also vector quantities. If r is a position vector of a particle at time t then $\frac{d\vec{r}}{dt}$ denotes its velocity.

3. Partial derivative is defined as

$$\frac{\partial f}{\partial x} = \lim_{\delta x \to 0} \frac{f(x + \delta x, y, z) - f(x, y, z)}{\delta x}$$

In case of partial derivative with respect to a variable, all the other remaining variables are taken as constant.

4. Vector differential operator del is denoted by ∇ and defined as

$$\nabla = \mathrm{i}\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

5. The gradient of a scalar function ∅ is defined as

grad $\emptyset = \nabla\emptyset = (\mathrm{i}\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z})\emptyset$

6. The divergence is dot product of del operator with any vector point function $\vec{f}$ and is given as

div. $\vec{f} = \nabla.\vec{f} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right).(if_x + jf_{y+}kf_z)$ where $\vec{f} = if_x + jf_{y+}kf_z$

$$= \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}$$

7. The curl of a vector $\vec{F} = F_x i + F_y j + F_j k$ is defined as

Curl $\overline{F} = \nabla \times \vec{F} = (i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}) \times (F_x i + F_y j + F_j k)$

8. The integral of a vector function $\vec{F}$ along a line or curve is called line integral and given as $\int_l \vec{F}.\vec{dl}$

9. If $\vec{F}$ is a vector function and s is a surface, then surface integral of a vector function $\vec{F}$ over the surface S is given as $\iint_s \vec{F}.d\vec{s}$

10. If dV denotes the volume defined by dxdydz then the volume integration of a vector F is defined as $\int_V F dV = \int_x \int_y \int_z F.dxdydz$

2.8 Glossary:

Displacement – net change in location of a moving body.

Differentiation- instantaneous rate of change of a function with respect to one of its variables
Integration- The process of finding a function from its derivative. (Reverse of differentiation)

Partial derivative- derivative of a function with respect to a variable, if all other remaining variables are considered as constant

Operator – An *Operator* is a symbol that shows a mathematical operation.

del operator - vector differentiation operator gradient- derivative of function.(rate of change of a function or slope)

divergence- rate at which density exits at a given region of space. (flux density)

Curl- describes the rotation of vector field.

line integral- Integration along a line.

surface integral- Integration along a surface.

volume integral- Integration along a volume.

1.9 **Self Assessment Question (SAQ):**

1. If $\emptyset(x, y, z) = (x^2 + y^2 + z^2)^{-5/2}$ then calculate $\nabla\emptyset$ at a point (1 1 1).
2. Calculate the gradient of a scalar function $\emptyset\,(x\ y\ z) = x^2 + y^2 + \ e^z$ at point (1, 2, -2).
3. If vector $\vec{B} = 3xyi + 5zj + 2yz^2k$ represents the magnetic field then calculate the flux at point (2, 2, 1).
4. Fiend the curl of a vector $\vec{A} = 3i + 5yzj + 5yz^2k.$
5. Given a vector function $\vec{F} = yi + xj$, calculate the line integration $\int_l \vec{F}.\vec{dl}$ from point (1,1,1) to (8, 2, -2) along the line joining these two points.
6. Show that $\nabla = 3y^4z^2i + 4x^3z^2j - 3x^2y^2k$ is a solenoidal vector.
7. Prove that $div\ grad\ \emptyset = \nabla^2\emptyset$
8. Prove that $div\ (\emptyset\ A) = \emptyset\ div\ A + A\ .\ grad\ \emptyset$
9. Explain the physical meaning of curl.
10. Explain different type of vector fields.

2.10 Reference Books:

1. Mechanics – D.S. Mathur, S Chand, Delhi
2. Concept of Physics- H C Verma, Bharti Bhawan, Patna
3. Physics Part-II, Robert Resnick and David Halliday, Wiley Eastern Ltd

2.11 Suggested readings:

1. Modern Physics, Beiser, Tata McGraw Hill
2. Fundamental University Physics-I, M. Alonslo and E Finn, Addition-Wesley Publication
3. Berkeley Physics Course Vol I, Mechanics, C Kittel et al, McGraw- Hill Company

2.12 Terminal questions:

2.12.1 Short answer type questions

1. Define gradient of a scalar function $\emptyset$.

2. Show that $\nabla\emptyset$ is a vector whose magnitude is equal to maximum rate of change of $\emptyset$ with respect to space variable.

3. Show that $\nabla\emptyset$ is perpendicular to surface $\emptyset$.

4. Solve $\nabla\left(\frac{1}{r}\right)$ for $r \neq 0$

5. If vector $\vec{F} = 6xzi - y^2j + yzk$ then calculate $\int_S \vec{F}.\hat{n}\, dS$ where S is the surface of a cube with boundaries $x = 0$ *to* $x = 2$, $y = 0$ *to* $y = 2$, $z = 0$ *to* $z = 2$.

6. Obtain the value $[grad\ \emptyset(\vec{r})] \times \vec{r}$

7. Find the area of parallelogram determined by the vectors $(i + 2j + 3k)$ and $(-3i - 2j + 4k)$.

Essay type questions

1. Define divergence of a vector function and its physical significance. Obtain the expression for the divergence of a vector $\vec{F}$.
2. Define curl of a vector function and its physical significance. Obtain the expression for the curl of a vector $\vec{F}$.
3. Prove that $\nabla \times (\vec{A} \times \vec{B}) = (\vec{B}.\vec{\nabla})\vec{A} - \overrightarrow{(A.\nabla)}\vec{B} + \vec{A}\, div\, \vec{B} - \vec{B}\, div \vec{A}$
4. Prove that any vector function can be expressed as the sum of lamellar vector and solenoidal vector.
5. Derive the equation of continuity

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\, J = 0$$

And show that how this equation express charge conservation.

6. Show that $\vec{u} \times \vec{v}$ is solenoidal if $\vec{u}$ and $\vec{v}$ are irrotational.

UNIT 3 ELASTICITY AND ELASTIC CONSTANT

Structure

3.1 Introduction:

In this unit 6 you will study the dynamics of rigid bodies. It means that during the motion of the body if the relative position of constituent particles remains same then the body is termed as **rigid body**. After the better understanding of rigid body you should try to understand the physical concept of non rigid body where the position of the constituent particles in the body changed after the application of external force.

In physics, a rigid body is an idealization of a solid body in which deformation is neglected. In other words, the distance between any two given points of a rigid body remains constant in time regardless of external forces exerted on it. Even though such an object cannot physically exist due to relativity, objects can normally be assumed to be perfectly rigid if they are not moving near the speed of light.

3.2 Objective:

The main objective of this unit is to study in detail about

A) The elasticity.

B) Physical interpretations of elasticity.

C) Elastic limit, stress, strain, Hooks law,

D) Different types of elastic constants and their interrelationships.

3.3 Elasticity:

A body's elastic properties refer to its capacity to withstand a deforming force or stress and to quickly regain its original size and shape when the stress has been removed. When these forces are released, an object made of an elastic material will revert to its original size and shape. Two different sorts of material parameters affect a material's degree of elasticity. The first kind of material parameter is referred to as a modulus, and it gauges how much stress (force per unit area) is required to produce a specific level of deformation. Modulus is measured in Pascal units (Pa).Typically, a greater modulus means that the material is more difficult to distort. The elastic limit is measured by the second category of parameter. The stress beyond which the material ceases to behave elastically and undergoes irreversible deformation may serve as the limit. The material will elastically rebound to a permanently deformed shape if the stress is relieved as opposed to its original shape.

3.3.1. Strain:

- When a body is under a system of forces or couples in equilibrium then a change is produced in the dimensions of the body.
- This fractional change or deformation produced in the body is called strain.
- Strain is a dimensionless quantity.
- Strain is of three types

(a) Longitudinal strain:- It is defined as the ratio of the change in length to the original length. If L is the original length and ΔL is the change in length then ΔL/L is termed as longitudinal strain. Experiments have shown that the change in length (ΔL) depends on only a few variables. As already noted ΔL is proportional to the applied force F and depends on the substance from which the object is made. Additionally, the change in length is proportional to the original length L0 and inversely proportional to the cross-sectional area of the wire or rod. For example, a long guitar string will stretch more than a short one, and a thick string will stretch less than a thin one.

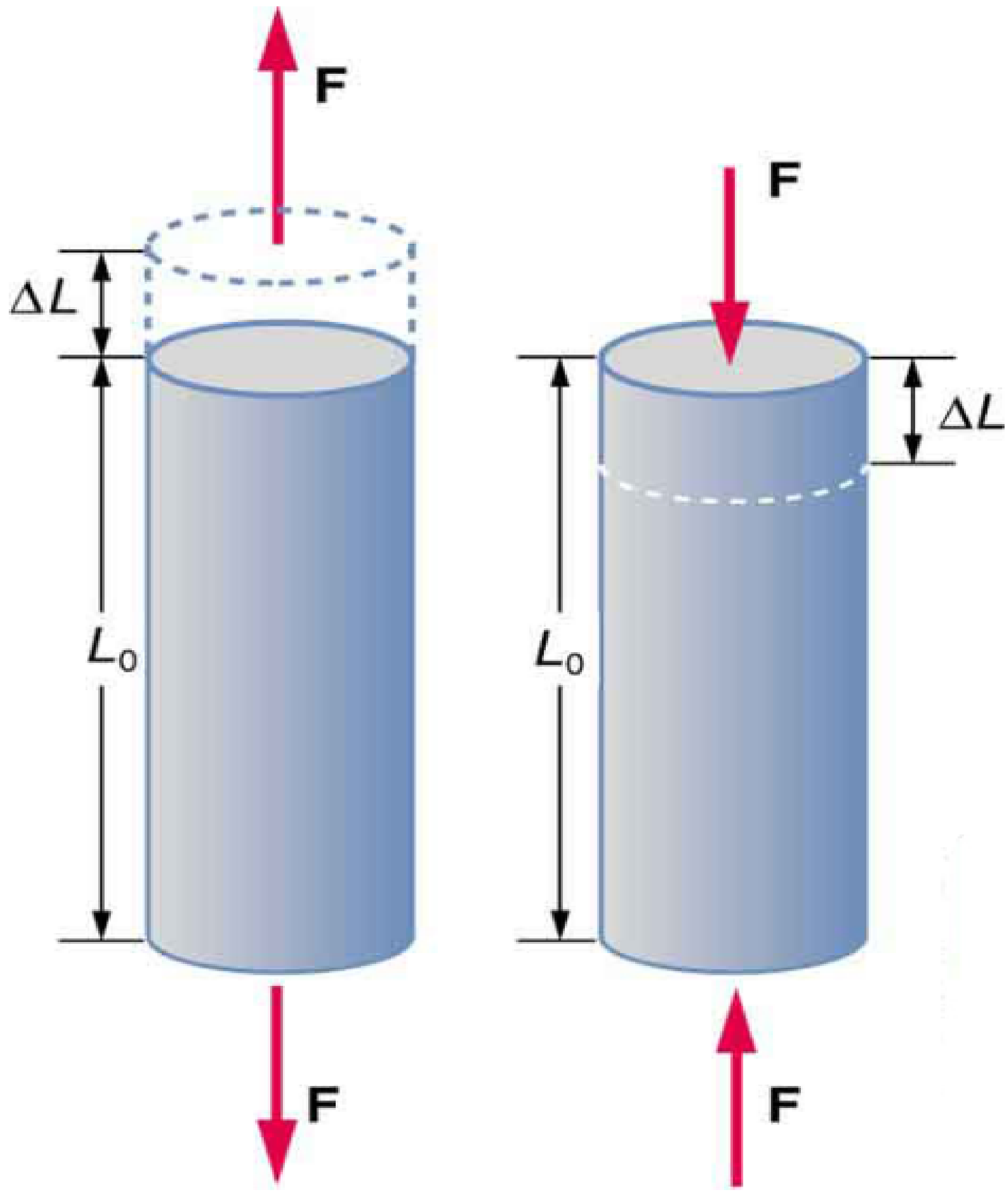


Figure 3.1

(b) Volume strain:-It is defined as the ratio of change in volume to the original volume. **(c) Shearing strain:-** If the deforming forces produce change in shape of the body then the strain is called shear strain. In practice since θ is much smaller than l so, tanθ≅ θ and the strain is simply the angle θ(measured in radians). Thus, shear strain is pure number without units as it is ratio of two lengths.

3.3.2 Stress:

When the external deforming forces act on a body, internal forces opposing the former are developed at each section of the body. The magnitude of the internal forces per unit area of the section is called **stress.** In the equilibrium state of the deformed body, the internal forces are equal and opposite of the external forces. Therefore, **Stress is measured by the external forces per unit area of their application.** The dimensions are $[ML^{-1} T^{-2}]$ and its units are N/m^2. The details are discussed in the different types of elastic constants. Stress is the force per unit area on a body that tends to cause it to change shape. Stress is a measure of the internal forces in a body between its particles. These internal forces are a reaction to the external forces applied on the body that cause it to separate, compress or slide. External forces are either surface forces or body forces. Stress is the average force per unit area that a particle of a body exerts on an adjacent particle, across an imaginary surface that separates them.

The formula for uniaxial normal stress is:

$$\sigma = \frac{F}{A}.$$

Where σ is the stress, F is the force and A is the surface area.In SI units, force is measured in Newtonand area in square meters.

3.3.3 *What is elastic limit:*

Elastic limit is the upper limit of deforming force up to which, if deforming force is removed, the body regains its original form completely and beyond which if deforming force is increased, the body loses its property of elasticity and gets permanently deformed. This mobile friendly simulation allows students to stretch and compress spring to explore relationships among force, spring constant, displacement and potential energy in a spring. You can use it to promote understanding of predictable mathematical relationship that underlies Hooke's law. Playing around with this simulation you can get an understanding of **restoring force**.

3.3.4 Stress- Strain curve:

In the curve given below, The part OA is a straight line which shows that up to the point A, stress is directional proportional to strain.i.e. Hooks law is obeyed up to A. The point A is called the limit of proportionality.

If the stress is further increased, a point B known as elastic limit of the material is reached. This point lies near the point A and up to this point; the wire takes back its original length, when the load is removed. Hence for the part AB of the curve, stress is necessarily proportional to strain.
On increasing the load beyond elastic limit, the stress-strain curve takes a bend. Now, at any point if the wire is unloaded, it does not regain its original length and gets permanent stretch, which we call **permanent set**. If the wire is now loaded, an entirely new stress-strain curve will represent its behavior.
If the load is further increased, a point C is reached, where the strain is much greater for a small increase in the load. This point C is called the **Yield Point** and the corresponding stress being the **yield stress.** Beyond this point C, the extension increases rapidly without an increase in the load, i.e. the material of the wire flows beyond C. This is known as **plastic flow**. As the wire becomes thin, the stress becomes considerably greater and the wire cannot support the same as before and wire is to be prevented from being broken, the load must be diminished.
After crossing the yield point, the thinning of wire no longer remains uniform and the diameter of a section decreases considerably. Now, the wire shows a phenomenon, known as 'necking'. Immediately, as this occur, the stress decreases automatically and the portion EF of the curve is obtained; ultimately a point F is known as **Breaking- point**, is reached, at which the wire breaks. The stress corresponding to this point is F is known as **breaking stress**. Here, the stress corresponding to E is called the **ultimate strength** or **tensile strength** of the given material. The tensile strength of the material is defined as the ratio of maximum load to which the specimen wire may be subjected by slowly increasing load to the original cross-sectional area of the wire. The ultimate strength of a material is, however measured by the stress causing the test specimen to break.

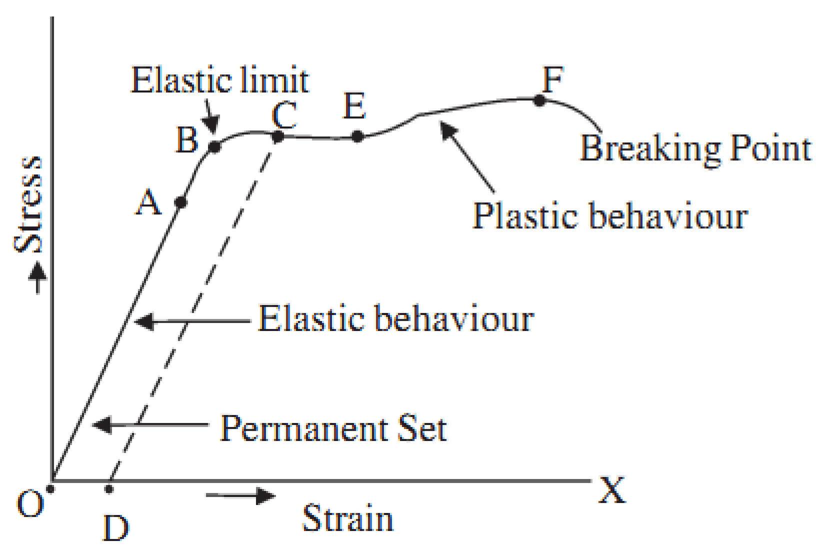


Figure 3.2

3.4 Hooke's Law:

Robert Hooke (1635-1703) invented the law as, within the proportional limit stress is directly proportional to strain. When studying springs and elasticity, the 17th century physicist Robert *Hooke* noticed that the stress vs strain curve for many materials has a linear region. Within certain limits, the force required to stretch an elastic object such as a metal spring is directly proportional to the extension of the spring.

- Hook's law is the fundamental law of elasticity and is stated as “for small deformations stress is proportional strain.
- stress /strain = constant
- This constant is known as modulus of elasticity of a given material, which depends upon the nature of the material of the body and the manner in which body is deformed.
- Hook's law is not valid for plastic materials.
- Units and dimension of the modulus of elasticity are same as those of stress.

Example for Understanding:

The spring is a marvel of human engineering and creativity. For one, it comes in so many varieties – the compression spring, the extension spring, the torsion spring, the coil spring, etc. – all of which serve different and specific functions. These functions in turn allow for the creation of many man-made objects, most of which emerged as part of the Scientific Revolution during the late 17th and 18th centuries.

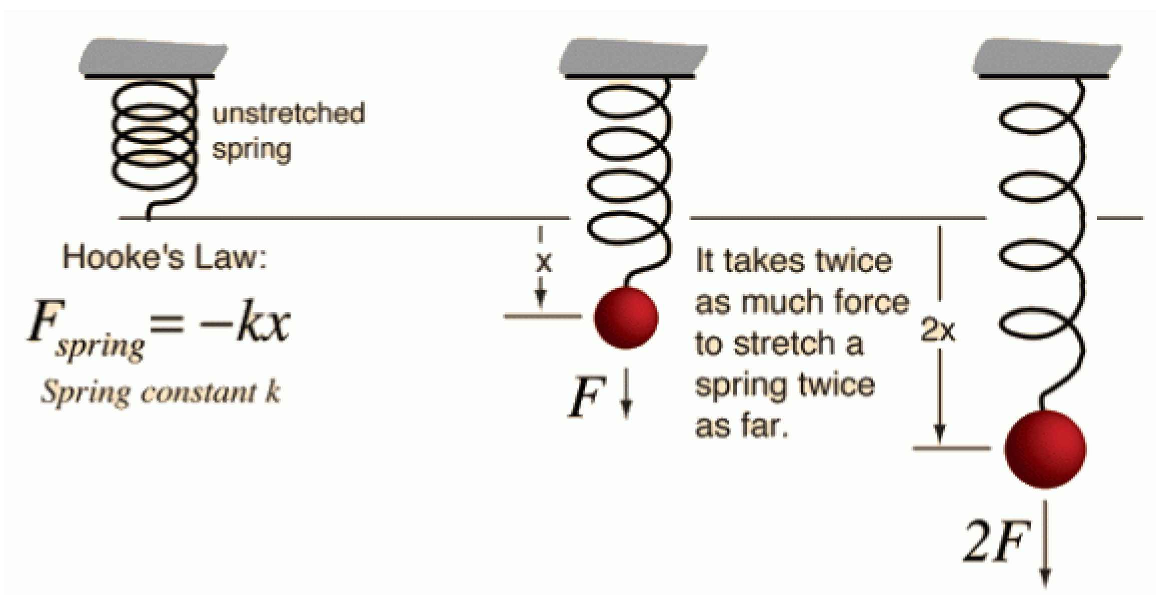


Figure 3.3: Illustration of Hooke’s Law, showing the relationship between force and distance when applied to a spring.

As an elastic object used to store mechanical energy, the applications for them are extensive, making possible such things as an automotive suspension systems, pendulum clocks, hand sheers, wind-up toys, watches, rat traps, digital micromirror devices, and of course, the slinky. Like so many other devices invented over the centuries, a basic understanding of the mechanics is required before it can so widely used.

This can be expressed mathematically as $F = -kX$, where F is the force applied to the spring (either in the form of strain or stress); X is the displacement of the spring, with a negative value demonstrating that the displacement of the spring once it is stretched; and k is the spring constant.

Hooke's law is the first classical example of an explanation of elasticity – which is the property of an object or material which causes it to be restored to its original shape after distortion. This ability to return to a normal shape after experiencing distortion can be referred to as a "restoring force". Understood in terms of Hooke's Law, this restoring force is generally proportional to the amount of "stretch" experienced.

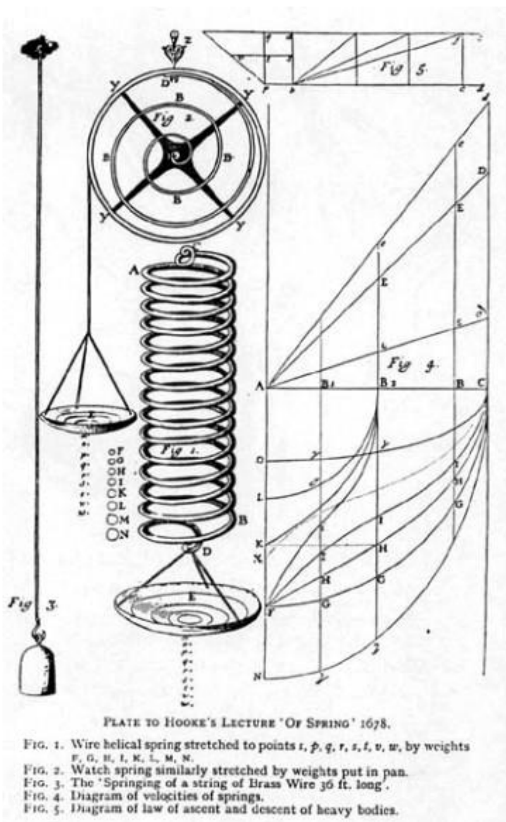


Figure 3.4 Illustration from Hooke's 1678 treaties "De potential restitutive (Of Spring)" Source: umn.edu

In addition to governing the behavior of springs, Hooke's Law also applies in many other situations where an elastic body is deformed. These can include anything from inflating a balloon and pulling on a rubber band to measuring the amount of wind force needed to make a tall building bend and sway.

This law has had many important practical applications, with one being the creation of a balance wheel, which made possible the creation of the mechanical clock, the portable timepiece, the spring scale and the manometer (the pressure gauge). Also, because it is a close approximation of all solid bodies (as long as the forces of deformation are small enough), numerous branches of science and engineering as also indebted to Hooke for coming up with this law. These include the disciplines of seismology, molecular mechanics and acoustics.

However, like most of the classical mechanics, Hooke's Law only works within a limited frame of reference. Because no material can be compressed beyond a certain minimum size (or stretched beyond a maximum size) without some permanent deformation or change of state, it only applies so long as a limited amount of force or deformation is involved. In fact, many materials will noticeably deviate from Hooke's law well before those elastic limits are reached.

Still, in its general form, Hooke's Law is compatible with Newton's laws of static equilibrium. Together, they make it possible to deduce the relationship between strain and stress for complex objects in terms of the intrinsic material properties. For example, one can deduce that a homogeneous rod with uniform cross section will behave like a simple spring when stretched, with a stiffness (*k*) directly proportional to its cross-section area and inversely proportional to its length.

3.5 Elastic Constants:

3.5.1 Young's Modulus of Elasticity:

- Young's Modulus of elasticity is the ratio of longitudinal stress to longitudinal strain, within elastic limit.
- It is denoted by Y.
- Young's Modulus of elasticity is given by

$$Y = \frac{longitudinal\ stress}{lingitudinal\ strain}$$

- Let us now consider a wire of length l having area of cross-section equal to A.

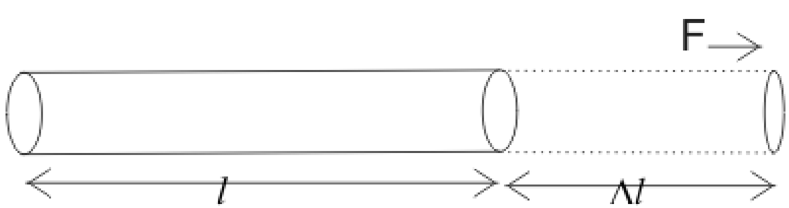


FIGURE 3.

Figure 3.4

If the force F acting on the wire, stretches the wire by length Δ l then

$$longitudinal\ stress = \frac{F}{A} \qquad (1)$$

and

$$lingitudinal\ strain = \frac{\Delta l}{l} \qquad (2)$$

From (1) and (2) we have Young's modulus of elasticity as

$$Y = \frac{Fl}{A\Delta l}$$

- Young's modulus of elasticity has dimensions of force/Area i.e. of pressure.
- Unit of Young's modulus is N/m^2.
- If area of cross-section of a wire is given by A = πr^2 then Young's modulus is

$$Y = \frac{Fl}{\pi r^2 \Delta l}$$

again if A = π r^2 = $1cm^2$ and Δ l = l = 1cm then Y = F

Thus, ***Young's modulus can also be defined as the force required to double the length of a wire of unit length and unit area of cross-section.***

3.5.2 Bulk Modulus of Elasticity:

- The ratio of normal stress to volume strain within elastic limits is called Bulk Modulus of elasticity of a given material.
- It is denoted by K.
- Suppose a force F is applied normal to a surface of a body having cross-sectional area equal to A, so as to cause a change in it's volume. If applied force bring about a change ΔV in the volume of the body and V is the original volume of the body then,

$$normal\ stress = \frac{F}{A}$$

and

Volume strain = ΔV/V

So, Bulk Modulus of elasticity would be,

$$Bulk\ modulus = K = \frac{normal\ stress}{volume\ strain}$$

Thus,

$$K = \frac{FV}{A\Delta V}$$

- For gases and liquids the normal stress is caused by change in pressure i. e., normal stress = change in pressure ΔP. Thus, bulk Modulus is

$$K = -\frac{V\Delta P}{\Delta V}$$

here negative sign indicates that the volume decreases if pressure increases and vice-versa.
- For extremely small changes in pressure and volume, the Bulk Modulus is given by

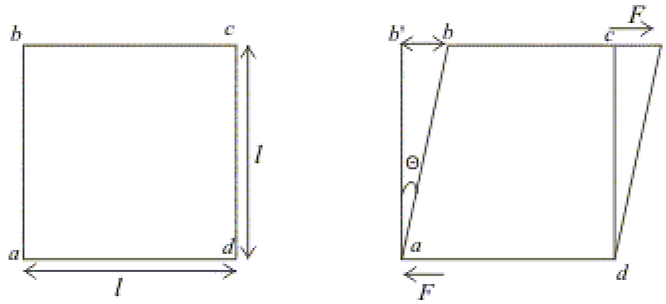
$$K = -V\frac{dP}{dV}$$

- Reciprocal of Bulk Modulus is called compressibility of the substance. Thus,

$$compressibility = \frac{1}{K} = \frac{\Delta V}{V\Delta P}$$

3.5.3 Modulus of Rigidity (η):

- When a body is sheared, the ratio of tangential stress to the shearing strain within elastic limits is called the Modulus of Rigidity and it is denoted by η.
- If lower face of the rectangular block shown below in the figure, is fixed and tangential force is applied at the upper face of area A, then shape of rectangular block changes.



A block subjected to shearing stress deforms by an angle θ (Front face shown)

Figure 3.6

So,

shearing strain = θ ≅ tanθ = $\frac{bb'}{l}$ = $\frac{x}{l}$ (where bb̓ = x, displacement of upper face)

or,

$$tengential\ stress = \frac{F}{A}$$

Thus,

$$modulus\ of\ rigidity = \eta = \frac{Fl}{xA}$$

3.5.4 Poisson's Ratio:

The ratio of lateral strain to linear strain is called Poisson's ratio. It is denoted by 'σ'. The lateral strain is defined as the ratio of change in diameter to original diameter. If a wire of length L and diameter D, is elongated by pulling to length (L+ l), it's literal dimension (diameter) decreases to (D-d).

σ = lateral strain/linear strain.

$$= \frac{(^{d}/_{D})}{(^{l}/_{L})}$$

The value of 'σ' varies from 1/3 to 1/4 depending upon the material. If τ is the applied tensile stress and γ is the young's modulus of the wire, then linear strain is (τ/γ) and lateral strain is $\sigma.(^{\tau}/_{\gamma})$

3.5.5 Points you must note about elastic modulus:

1. The value of elastic modulus is independent of stress and strain. It depends only on the nature of the material.
2. Greater value of modulus of elasticity means that the material has more elasticity i.e., material is more elastic.
3. Young's Modulus and Shear Modulus exists only for solids while Bulk Modulus is defined for all three stats of matter.
4. Three modulus of elasticity Y, η and K depends on temperature. Their value decreases with the increase in temperature.
5. In case of longitudinal stress, shape remains unchanged while the volume changes. In tensile one volume increases while in compressive one volume decreases.
6. In shear stress, volume remains the same but shape changes.
7. In volume stress, volume changes but shape remains the same.

3.6 Relation between elastic constants

3.6.1 Relation between Y, η and σ:

$$Y = 2\eta\,[1+\sigma]$$

3.6.2 Relation between Y, σ and K:

$$\mathbf{Y} = 3K\,(1-2\sigma)$$

3.6.3 Relation between Y, η and K:

$$Y = 9K\eta/\,(3K+\eta)$$

3.7 DERIVATION OF RELATION AMONG ELASTIC CONSTANTS:

3.7.1 Derivation for the Relation between Y, η and σ:

Let us establish a relation among the elastic constants Y, η and σ. Consider a cube of material of side 'a' subjected to the action of the shear and complementary shear stresses as shown in the figure and producing the strained shape as shown in the figure below. Assuming that the strains are small and the angle A C B may be taken as 45^0.

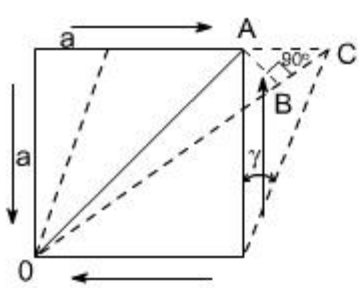


Figure 3.7

Therefore, strain on the diagonal OA = Change in length / original length

Since angle between OA and OB is very small hence OA = OB therefore BC, is the change in the length of the diagonal OA

Thus, strain on diagonal OA = BC/OA

$= AC \cos 45^0 / OA$

$OA = a/ \sin 45^0$

$= a. \sqrt{2}$

Hence, tensile strain = $AC/a\sqrt{2} \ . \ 1/\sqrt{2}$

$\gamma = AC / 2a$

but $AC = a\,\gamma$ (as $\tan \gamma \cong \gamma = AC/a$) , where γ = shear strain

Thus, the strain on diagonal = $ar/2a = r/2$

From the definition, If τ is the applied sharing stress, then

$\eta = \tau / \gamma$ or

$\gamma = \tau / \eta$

Thus, the strain on the diagonal= $\gamma/2 = \tau / 2\eta$.

Now this shear stress system is equivalent or can be replaced by a system of direct stresses at 45^0 as shown below. One set will be compressive, the other tensile, and both will be equal in value to the applied shear strain stress τ, thus decreasing diagonal DN and increasing OC in length.

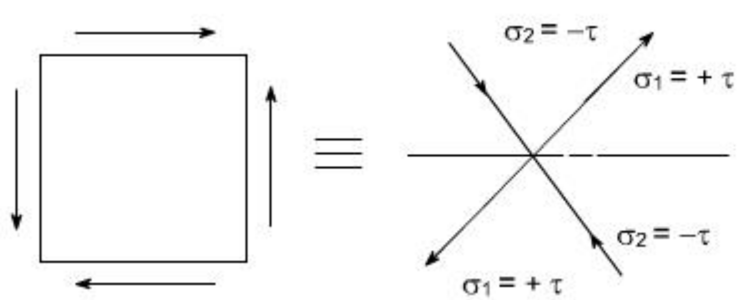


Figure 3.8

Thus, for the direct state of stress system which applies along the diagonals:

strain on diagonal = $\sigma_1 = \sigma_1/Y - \sigma.\ \sigma_2/Y$, (a compressive strain τ_2/γ is equivalent to a tensile strain $\tau \times (\tau_2/\gamma)$ in the lateral direction OA, increasing it's length)

$=\tau/ Y- \sigma. (-\tau)/Y$

$= \tau/ Y(1+\sigma).$

Equating the two strains one may get

$$\tau/2\eta= \tau(1+\sigma)/ Y$$

$$Y= 2\eta(1+\sigma).$$

We have introduced a total of four elastic constants, i.e Y, η, K and σ. It turns out that not all of these are independent of the others. In fact given any two of them, the other two can be found.

Again $\quad Y= 3K(1-2\sigma)$

$$K = Y/\{3(1-2\sigma)\}$$

When σ = 0.5, the value of E is infinite, rather than a zero value of E and volumetric strain is zero, or in other words, the material is incompressible.

3.7.2 Derivation for the Relation between Y, K and σ:

Consider a cube subjected to three equal stresses τ as shown in the figure below, due to which it is being expanded in all directions.

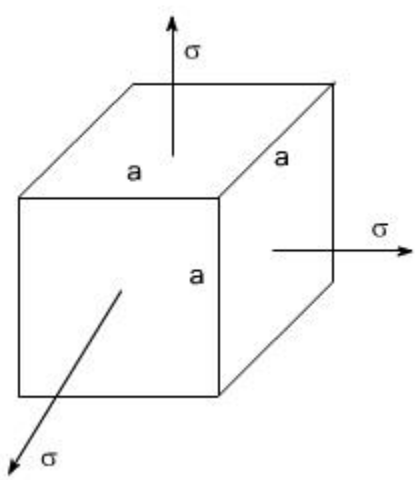


Figure 3.9

The total strain in one direction or along one edge due to the application of hydrostatic stress or volumetric stress τ is given as;

Liner strain along one edge =

$\tau/Y - \sigma.\tau/Y - \sigma.\tau/Y$

$\tau\ (1- 2\sigma) / Y$

volumetric strain = 3 times of linear strain

$= 3.\tau\ (1- 2\sigma)/ Y.$

By definition

Bulk modulus of elasticity (K) = Volumetric stress / Volumetric strain

or

Volumetric strain = τ / K

Equating the two strains we get

$Y= 3K(1-2\sigma)$

3.7.3 Derivation for the Relation between Y, η and K:

The relationship between Y, η and K can be easily determined by eliminating σ from the already derived relations

$Y = 2\eta(1 + \sigma)$ and $Y = 3K(1 -2\sigma)$.

Thus, the following relationship may be obtained

$Y = 9K\eta/ (3K+\eta)$.

3.7.4 Derivation for the Relation between η, K and σ:

From the already derived relations, Y can be eliminated

$Y = 2\eta [1+ \sigma]$

$Y = 3K (1-2\sigma)$

Thus, we get

$2\eta [1+ \sigma] = 3K (1-2\sigma)$

$\sigma= 0.5(3K-2\eta)(\eta+3K)$.

3.8 Summary:

In this unit, you have studied about elastic materials and their elastic properties. To present the clear understanding of elasticity and different elastic constants, the elastic limit, Hooke's law and enter relationships between elastic constants have been discussed. The derivations of elastic constants in a different way are given in this chapter. Applications of elasticity in the field of science and technology have been described. The pictorial understanding of elastic constants such as Young modulus Y, Bulk modulus K and Modulus of rigidity η have been discussed.

3.9 Glossary:

Elastic – Regain – return into original shape

Elastic limit – a region to follow Hooke's law, stress proportional to strain

Stress – force per unit area – pressure

Limit- within the defined range

Shear – some deformation from original one

Confined- restricted

Undergo- suffer

Maintain- sustain

Resist- refuse to go along with

Strain – ratio of change in dimension to original dimension

Compressibility – reciprocal of bulk modulus

3.10 Terminal Questions:

3.10.1 Multiple Choice Questions:

1. Maximum limit up to which stress is applied on body without deformation is called

a. limit
b. elastic limit
c. strain
d. none of above

2. If a 1 m long steel wire having area 5 x 10^{-5} is stretched through 1 mm by force of 10, 000 N then young modulus of wire is

a. 2 x 10^{11} N m^{-2}
b. 3 N m^{-4}
c. 4 N m^{-2}
d. 5 N m^{-2}

3. Ratio of stress to strain is

a. 1
b. 0
c. 3
d. constant

4. Stress is

a. External force
b. Internal resistive force
c. Axial force
d. Radial force

5. Which of the following is not a basic type of strain?

a. Compressive strain
b. Shear strain
c. Area strain
d. Volume strain

6. Tensile Strain is

a. Increase in length / original length
b.Decrease in length / original length
c. Change in volume / original volume
d. All of the above

7. Compressive Strain is

a. Increase in length / original length
b. Decrease in length / original length
c. Change in volume / original volume
d. All of the above

8. Hooke's law is applicable within

a. Elastic limit
b. Plastic limit
c. Fracture point
d. Ultimate strength

9. Young's Modulus of elasticity is

a. Tensile stress / Tensile strain

b. Shear stress / Shear strain
c. Tensile stress / Shear strain
d. Shear stress / Tensile strain

10. Maximum limit up to which stress is applied on body without deformation is called

a. limit
b. elastic limit
c. strain
d. none of above.

(Ans: 1-b, 2-a, 3-d, 4-b, 5-c, 6-a, 7-b, 8-a, 9-a, 10-b)

1. If a 1 m long steel wire having area 5 * 10^{-5} is stretched through 1 mm by force of 10, 000 N then young modulus of wire is

a. 2X 10^{11} N m^{-2}
b. 3 N m^{-4}
c. 4 N m^{-2}
d. 5 N m^{-2}

2. Ratio of stress to strain is

a. 1
b. 0
c. 3
d. constant

3. Which of the following material is more elastic?

(a) Rubber

(b) Glass

(c) Steel

(d) Wood

4. A load of 1 kN acts on a bar having cross-sectional area 0.8 cm^2 and length 10 cm. The stress developed in the bar is

(a) 12.5 N/mm^2

(b) 25 N/mm^2

(c) 50 N/mm^2

(d) 75 N/mm^2

5. A brittle material has

(a) No elastic zone

(b) No plastic zone

(c) Large plastic zone

(d) None of these

6. The length of a wire is increased by 1 mm on the application of a certain load. In a wire of the same material but of twice the length and half the radius, the same force will produce an elongation of

(a) 0.5 mm

(b) 2 mm

(c) 4 mm

(d) 8 mm.

3.10.2 Solved Problems:

Question 1. A block of gelatin is 60 mm by 60 mm by 20 mm when unstressed. A force of .245 N is applied tangentially to the upper surface causing a 5 mm displacement relative to the lower surface. The block is placed such that 60X60 comes on the lower and upper surface. Find the shearing stress, shearing strain and shear modulus

(a) (68.1 N/m^2 , .25 , 272.4 N/m^2)
(b) (68 N/m^2 , .25 , 272 N/m^2)
(c) (67 N/m^2 , .26 , 270.4 N/m^2)
(d) (68.5 N/m^2 , .27 , 272.4 N/m^2)

Solution:
Shear stress $=F/A=.24536\times10^{-4}=68.1N/m^2$

Shear strain$=\tan\theta=d/h=5/20=.25$

Shear modulus

strain=272.4N/m

Question 2.A steel wire of diameter 4 mm has a breaking strength of $4X10^5$N. The breaking strength of similar steel wire of diameter 2 mm is

(a)$1X10^5$N.

(b) $4X10^5$N.
(c) $16X10^5$N.
(d) none of the these

Solution

Breaking strength is proportional to square of diameter,Since diameter becomes half,Breaking strength reduced by 1/4. Hence A is correct.

Question 3.What is the SI unit of modulus of elasticity of a substance?

(a) Nm^{-1}
(b) Nm^{-2}
(c) Jm^{-1}
(d) Unit less quantity

Solution

Answer is b

Question 4A thick uniform rubber rope of density 1.5 gcm^{-3} and Young Modulus $5X1010^6$ Nm^{-2} has a length 8 m. when hung from the ceiling of the room, the increase in length due to its own weight would be ?

(a) .86m
(b) .2m
(c) .1m
(d) .096m

Solution The weight of the rope can be assumed to act at its mid point. Now the extension x is proportional to the original length L. if the weight of the rope acts at its midpoint, the extension will be that produced by the half of the rope. So replacing L by L2L2 in the expression for Young 's Modulus. Substituting the values, we get
l=.096ml=.096m

3.11 Numerical Questions:

Problem 1: A spring stretches 5 cm when a load of 20 N is hung on it. If instead, we put a load of 30 N, how much will the spring stretch? What is the spring constant?

Solution: There are a couple of ways to solve this problem.
Way #1: Notice that 30 N = 20 N + 10 N
20 N creates a stretch of 5 cm. Since 10 N is half of 20 N, then 10 N will create a stretch that is half of 5 cm or 2.5 cm.
Total stretch = 5 cm + 2.5 cm = 7.5 cm
Way #2: Set up a proportion.

5 cm is to 20 N as 'new stretch' is to 30 N.
5 × 30 = new stretch × 20
150 = new stretch × 20
new stretch = 7.5 cm
To get the spring constant, make a couple of good observation.
20 = 4 × 5
30 = 4 × 7.5
F = 4 × x
F is the force applied and x is the stretch
The spring constant is k = 4

Problem 2: With a weight of 25 kg, a spring stretches 6 cm. Its elastic limit is reached with a weight of 150 kg. How far did the spring stretch?
Since 150 kg divided by 25 kg = 6 kg, 150 kg is 6 times bigger.
The stretch will then be 6 times bigger than 6 cm or 36 cm.

Problem 3: A spring has a spring constant that is equal to 3.5. What force (in kilograms) will make it stretch 4 cm?
F = k × x
F = 3.5 × 4
F = 14 kg

Problem 4: When the weight hung on a spring is increased by 60 N, the new stretch is 15 cm more. If the original stretch is 5 cm, what is the original weight?
We will need some algebra and a proportion to solve this tough word problem.
Let x be the original weight, then x + 60 is the new weight
If the original stretch is 5 cm, then the new stretch is 20 cm.

x × 20 = 5 × x + 5 × 60

20x = 5x + 300

15x = 300

Since 15 × 20 = 300, the original weight is 20 N.

3.12 References:

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3.13 Suggested Readings:

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3. Physics Part-I, Robert Resnick and David Halliday, Wiley Eastern Ltd
4. Berkeley Physics Course Vol I, Mechanics, C Kittel et al, McGraw- Hill Company

UNIT 4 SURFACE TENSION

Structure

4.1 Introduction:

Surface tension is essentially the tendency of liquids to contract in order to provide the least amount of surface area. The attraction between each liquid molecule, which tries to stick together and close the space between them, is what causes this.

Liquids interacting with gases or solids have the same force of attraction. In fact, this is the reason why water frequently clings to a beaker's sides even after being poured out. Or perhaps a coating of oil forms on the water's surface.

We must first comprehend surface energy in order to comprehend surface tension. The effort put forth per unit area to produce and maintain a surface is known as surface energy. This is a result of the liquid's molecules' strong attraction to one another.

Let's use a surface tension example to better grasp this. Insects of any size that you may see in water do not perish. This is due to the fact that their weight is insufficient to overcome the mutual force between the water molecules on the surface. As a result, when an insect moves, it cannot produce enough surface energy to release the water's surface tension.

Surface tension in water is caused by the cohesion of the water molecules. Water atoms adhere to one another. However, as one surface comes into contact with air, an even stronger link is created between the molecules present on this surface layer since there are less water molecules to cling to on the surface. As a result, there is now a significant surface tension that forms a film between the air and the water.

4.2 Objective:

The main objective of this unit is to study in detail about the surface tension and their physical interpretations and also to acquaint the student about the elastic limit, stress, strain, Hooks law, different types of elastic constants and their interrelationships. However, the deformation in the object can occur after the application of external force. The physical interpretation and their concerned relations have been interpreted in their subsequent sections. At last the varities of problems related to different topics have been discussed for better understanding.

4.3. Surface tension:

Surface tension is the tendency of liquid surfaces at rest to shrink into the minimum surface area possible. Surface tension is what allows objects with a higher density than water such as razor blades and insects (e.g. water strides) to float on a water surface without becoming even partly submerged.

Surface tension is the energy, or work, required to increase the surface area of a liquid due to intermolecular forces. Since these intermolecular forces vary depending on the nature of the liquid (e.g. water vs. gasoline) or solutes in the liquid (e.g. surfactants like detergent), each solution exhibits differing surface tension properties. Whether you know it or not, you already have seen surface tension at work. Whenever you fill a glass of water too far, you may notice afterward that the level of the water in the glass is actually higher than the height of the glass. You may have also noticed that the water that you spilled has formed into pools that rise up off the counter. Both of these phenomena are due to surface tension.

At liquid air interfaces, surface tension results from the greater attraction of liquid molecules to each other (due to cohesion) than to the molecules in the air (due to adhesion).

There are two primary mechanisms in play. One is an inward force on the surface molecules causing the liquid to contract. Second is a tangential force parallel to the surface of the liquid. This *tangential* force is generally referred to as the surface tension. The net effect is the liquid behaves as if its surface were covered with a stretched elastic membrane. But this analogy must not be taken too far as the tension in an elastic membrane is dependent on the amount of deformation of the membrane while surface tension is an inherent property of the liquid air or liquid vapour interface.

Because of the relatively high attraction of water molecules to each other through a web of hydrogen bonds, water has a higher surface tension (72.8 millinewtons (mN) per meter at 20 °C) than most other liquids. Surface tension is an important factor in the phenomenon of capillarity.

Surface tension has the dimension of force per unit lenth, or of energy per unit area. The two are equivalent, but when referring to energy per unit of area, it is common to use the term surface energy which is a more general term in the sense that it applies also to solids.

In material sciences, surface tension is used for either surface stress or surface energy.

4.3.1. Causes

Due to the cohesive forces, a molecule located away from the surface is pulled equally in every direction by neighboring liquid molecules, resulting in a net force of zero. The molecules at the surface do not have the *same* molecules on all sides of them and therefore are pulled inward. This creates some internal pressure and forces liquid surfaces to contract to the minimum area.

There is also a tension parallel to the surface at the liquid-air interface which will resist an external force, due to the cohesive nature of water molecules.

The forces of attraction acting between molecules of the same type are called cohesive forces, while those acting between molecules of different types are called adhesive forces. The balance between the cohesion of the liquid and its adhesion to the material of the container determines the degree of wetting, the contact angle, and the shape of meniscus. When cohesion dominates (specifically, adhesion energy is less than half of cohesion energy) the wetting is low and the meniscus is convex at a vertical wall (as for mercury in a glass container). On the other hand, when adhesion dominates (when adhesion energy is more than half of cohesion energy) the wetting is high and the similar meniscus is concave (as in water in a glass).

Surface tension is responsible for the shape of liquid droplets. Although easily deformed, droplets of water tend to be pulled into a spherical shape by the imbalance in cohesive forces of the surface layer. In the absence of other forces, drops of virtually all liquids would be approximately spherical. The spherical shape minimizes the necessary "wall tension" of the surface layer according to Laplace law.

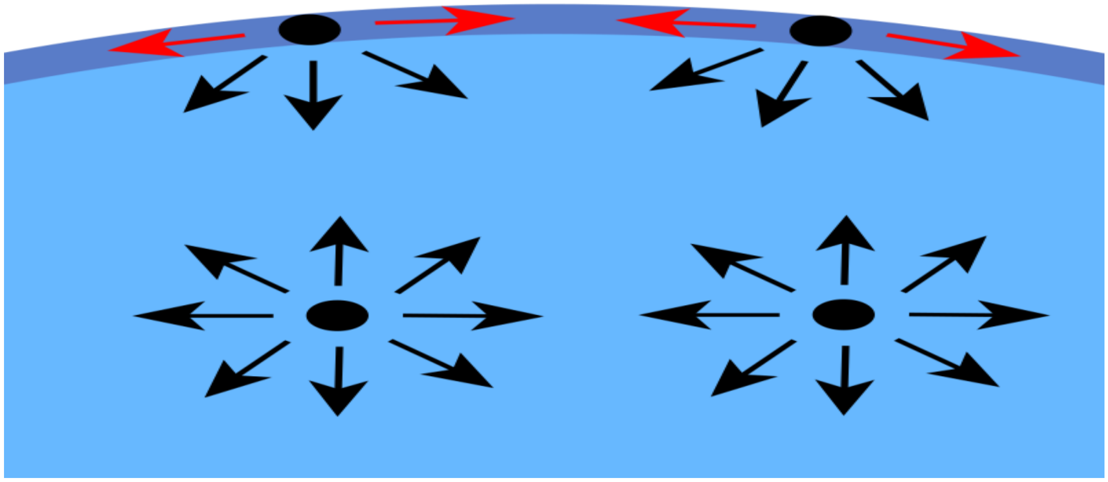

Fig. 4.1.: Diagram of the cohesive forces on molecules of a liquid

Another way to view surface tension is in terms of energy. A molecule in contact with a neighbor is in a lower state of energy than if it were alone. The interior molecules have as many neighbors as they can possibly have, but the boundary molecules are missing neighbors (compared to interior molecules) and therefore have higher energy. For the liquid to minimize its energy state, the number of higher energy boundary molecules must be minimized. The minimized number of boundary molecules results in a minimal surface area. As a result of surface area minimization, a surface will assume a smooth shape.

4.3.2. Molecular Perspective

In a sample of water, there are two types of molecules. Those that are on the outside, exterior, and those that are on the inside, interior. The interior molecules are attracted to all the molecules around them, while the exterior molecules are attracted to only the other surface molecules and to those below the surface. This makes it so that the energy state of the molecules on the interior is much lower than that of the molecules on the exterior. Because of this, the molecules try to maintain a minimum surface area, thus allowing more molecules to have a lower energy state. This is what creates what is referred to as surface tension. An illustration of this can be seen in Figure

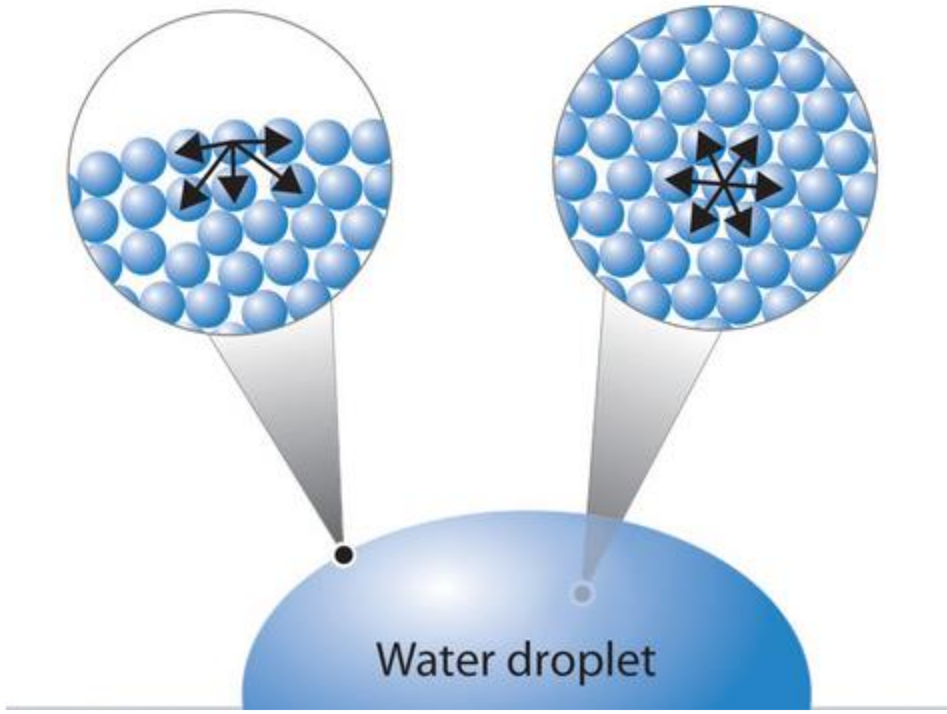

Fig. 4.2.: Water's Polar Property

The water molecules attract one another due to the water's polar property. The hydrogen ends, which are positive in comparison to the negative ends of the oxygen cause water to "stick" together. This is why there is surface tension and takes a certain amount of energy to break these intermolecular bonds. Same goes for other liquids, even hydrophobic liquids such as oil. There are forces between the liquid such as Van der Waals forces that are responsible for the intermolecular forces found within the liquid. It will then take a certain amount of energy to break these forces, and the surface tension. Water is one liquid known to have a very high surface tension value and is difficult to overcome.

Surface tension of water can cause things to float which are denser than water, allowing organisms to literally walk on water (Figure 2). An example of such an organism is the water strider, which can run across the surface of water, due to the intermolecular forces of the molecules, and the force of the strider which is distributed to its legs. Surface tension also allows for the formation of droplets that we see in nature.

Physical units : Surface tension, represented by the symbol T, is measured in force per unit lenth. Its SI unit is newton per meter but the cgs unit of dyne per centimeter is also used.

4.3.3. Definition: Surface tension can be defined in terms of force or energy.

In terms of force: Surface tension *T* of a liquid is the force per unit length. In the illustration on the right, the rectangular frame, composed of three unmovable sides (black) that form a "U" shape, and a fourth movable side (blue) that can slide to the right. Surface tension will pull the blue bar to the left; the force *F* required to hold the movable side is proportional to the length *L* of the immobile side. Thus the ratio *F*/*L* depends only on the intrinsic properties of the liquid (composition, temperature, etc.), not on its geometry. For example, if the frame had a more complicated shape, the ratio *F*/*L*, with *L* the length of the movable side and *F* the force required to stop it from sliding, is found to be the same for all shapes. We therefore define the surface tension as

$$\gamma = \frac{1}{2} \cdot \frac{F}{L}$$

γ = surface tension

F = force

L = length

The reason for the 1/2 is that the film has two sides (two surfaces), each of which contributes equally to the force; so the force contributed by a single side is $\gamma L = F/2$.

In terms of energy: Surface tension *T* of a liquid is the ratio of the change in the energy of the liquid to the change in the surface area of the liquid (that led to the change in energy). This can be easily related to the previous definition in terms of force if *F* is the force required to stop the side from *starting* to slide, then this is also the force that would keep the side in the state of *sliding at a constant speed* (by Newton's Second Law). But if the side is moving to the right (in the direction the force is applied), then the surface area of the stretched liquid is increasing while the applied force is doing work on the liquid. This means that increasing the surface area increases the energy of the film. The work done by the force *F* in moving the side by distance Δx is $W = F\Delta x$; at the same time the total area of the film increases by $\Delta A = 2L\Delta x$ (the factor of 2 is here because the liquid has two sides, two surfaces). Thus, multiplying both the numerator and the denominator of $\gamma = 1/2F/L$ by Δx, we get

This work W is, by the usual argument, interpreted as being stored as potential energy. Consequently, surface tension can be also measured in SI system as joules per square meter and in the cgs system as ergs per cm^2. Since mechanical system try to find a state of minimum potential energy, a free droplet of liquid naturally assumes a spherical shape, which has the minimum surface area for a given volume. The equivalence of measurement of energy per unit area to force per unit length can be proven by dimensional analysis.

4.3.4. Effects

4.3.4.1. Water

Several effects of surface tension can be seen with ordinary water:

A. Beading of rain water on a waxy surface, such as a leaf. Water adheres weakly to wax and strongly to itself, so water clusters into drops. Surface tension gives them their near-spherical shape, because a sphere has the smallest possible surface area to volume ratio..
B. Formation of drops occurs when a mass of liquid is stretched. The animation (below) shows water adhering to the faucet gaining mass until it is stretched to a point where the surface tension can no longer keep the drop linked to the faucet. It then separates and surface tension forms the drop into a sphere. If a stream of water were running from the faucet, the stream would break up into drops during its fall. Gravity stretches the stream, then surface tension pinches it into spheres.
C. Flotation of objects denser than water occurs when the object is nonwettable and its weight is small enough to be borne by the forces arising from surface tension. For example, water striders use surface tension to walk on the surface of a pond in the following way. The non-wettability of the water strider's leg means there is no attraction between molecules of the leg and molecules of the water, so when the leg pushes down on the water, the surface tension of the water only tries to recover its flatness from its deformation due to the leg. This behavior of the water pushes the water strider upward so it can stand on the surface of the water as long as its mass is small enough that the water can support it. The surface

of the water behaves like an elastic film: the insect's feet cause indentations in the water's surface, increasing its surface area and tendency of minimization of surface curvature (so area) of the water pushes the insect's feet upward.

D. Separation of oil and water (in this case, water and liquid wax) is caused by a tension in the surface between dissimilar liquids. This type of surface tension is called "interface tension", but its chemistry is the same.

E. Tears of wineis the formation of drops and rivulets on the side of a glass containing an alcoholic beverage. Its cause is a complex interaction between the differing surface tensions of water and ethanol it is induced by a combination of surface tension modification of water by ethanol together with ethanol evaporating faster than water.

4.3.4.2. Surfactants

Surface tension is visible in other common phenomena, especially when surfactants are used to decrease it:

- Soap bubbles have very large surface areas with very little mass. Bubbles in pure water are unstable. The addition of surfactants, however, can have a stabilizing effect on the bubbles. Surfactants actually reduce the surface tension of water by a factor of three or more.
- Emulsions are a type of colloid in which surface tension plays a role. Tiny fragments of oil suspended in pure water will spontaneously assemble themselves into much larger masses. But the presence of a surfactant provides a decrease in surface tension, which permits stability of minute droplets of oil in the bulk of water (or vice versa).

4.3.4.3. Floating objects

When an object is placed on a liquid, its weight F_w depresses the surface, and if surface tension and downward force become equal then it is balanced by the surface tension forces on either side F_s, which are each parallel to the water's surface at the points where it contacts the object. Notice that small movement in the body may cause the object to sink. As the angle of contact decreases, surface tension decreases. The horizontal components of the two F_s arrows point in opposite directions, so they cancel each other, but the vertical components point in the same direction and therefore add up to balance F_w. The object's surface must not be wettable for this to happen, and its weight must be low enough for the surface tension to support it. If *m* denotes the mass of the needle and *g* acceleration due to gravity, we have

$$F_w = 2F_s \sin\theta \quad \Leftrightarrow \quad mg = 2\gamma L \sin\theta$$

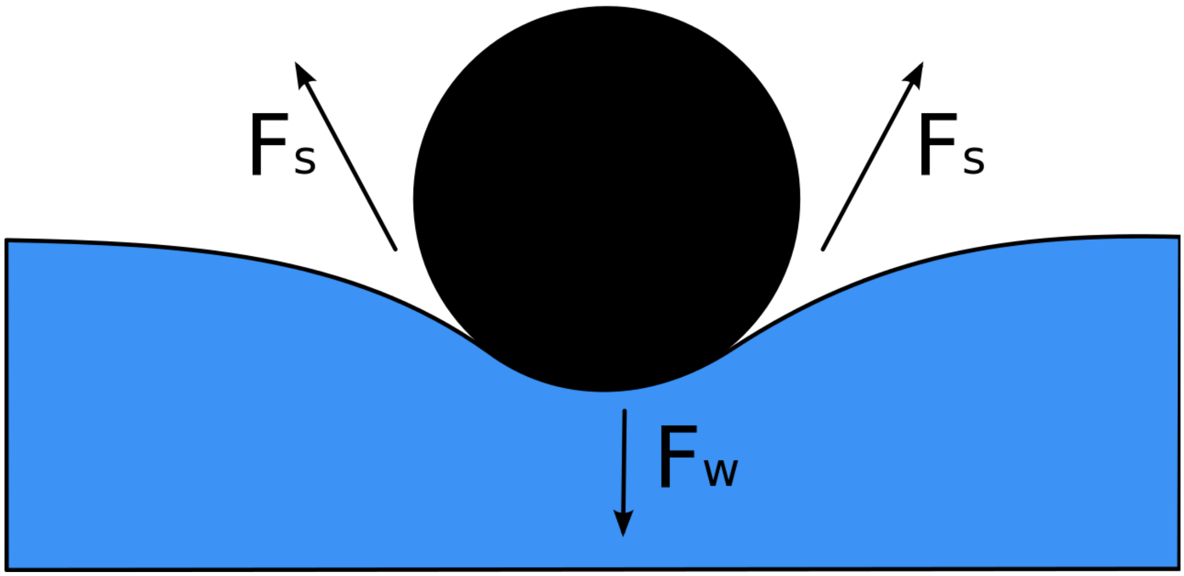

Fig. 4.3: Needle Floating on the Surface of Water

Cross-section of a needle floating on the surface of water. F_w is the weight and F_s are surface tension resultant forces.

4.3.4.4. Liquid surface

To find the shape of the minimal surface bounded by some arbitrary shaped frame using strictly mathematical means can be a daunting task. Yet by fashioning the frame out of wire and dipping it in soap-solution, a locally minimal surface will appear in the resulting soap-film within seconds.

The reason for this is that the pressure difference across a fluid interface is proportional to the mean curvature, as seen in the Young Laplace equation. For an open soap film, the pressure difference is zero, hence the mean curvature is zero, and minimal surfaces have the property of zero mean curvature.

4.3.4.5. Contact Angles

The surface of any liquid is an interface between that liquid and some other medium. The top surface of a pond, for example, is an interface between the pond water and the air. Surface tension, then, is not a property of the liquid alone, but a property of the liquid's interface with another medium. If a liquid is in a container, then besides the liquid/air interface at its top surface, there is also an interface between the liquid and the walls of the container. The surface tension between the liquid and air is usually different (greater) than its surface tension with the walls of a container. And where the two surfaces meet, their geometry must be such that all forces balance.

4.3.4.6. Special Contact Angles

Observe that in the special case of a water–silver interface where the contact angle is equal to 90°, the liquid–solid/solid–air surface tension difference is exactly zero.

4.3.4.7. Liquid in a Vertical Tube

An old style mercury barometer consists of a vertical glass tube about 1 cm in diameter partially filled with mercury, and with a vacuum (called Torricelli's vacuum) in the unfilled volume (see diagram to the right). Notice that the mercury level at the center of the tube is higher than at the edges, making the upper surface of the mercury dome-shaped. The center of mass of the entire column of mercury would be slightly lower if the top surface of the mercury were flat over the entire cross-section of the tube. But the dome-shaped top gives slightly less surface area to the entire mass of mercury. Again the two effects combine to minimize the total potential energy. Such a surface shape is known as a convex meniscus.

We consider the surface area of the entire mass of mercury, including the part of the surface that is in contact with the glass, because mercury does not adhere to glass at all. So the surface tension of the mercury acts over its entire surface area, including where it is in contact with the glass. If instead of glass, the tube was made out of copper, the situation would be very different. Mercury aggressively adheres to copper. So in a copper tube, the level of mercury at the center of the tube will be lower than at the edges (that is, it would be a concave meniscus). In a situation where the liquid adheres to the walls of its container, we consider the part of the fluid's surface area that is in contact with the container to have *negative* surface tension. The fluid then works to maximize the contact surface area. So in this case increasing the area in contact with the container decreases rather than increases the potential energy. That decrease is enough to compensate for the increased potential energy associated with lifting the fluid near the walls of the container.

If a tube is sufficiently narrow and the liquid adhesion to its walls is sufficiently strong, surface tension can draw liquid up the tube in a phenomenon known as capillary action. The height to which the column is lifted is given by Jurin law.

$$h = \frac{2\gamma_{\mathrm{la}} \cos\theta}{\rho g r}$$

where

- h is the height the liquid is lifted,
- γ_{la} is the liquid–air surface tension,
- ρ is the density of the liquid,
- r is the radius of the capillary,
- g is the acceleration due to gravity,
- θ is the angle of contact described above. If θ is greater than 90°, as with mercury in a glass container, the liquid will be depressed rather than lifted.

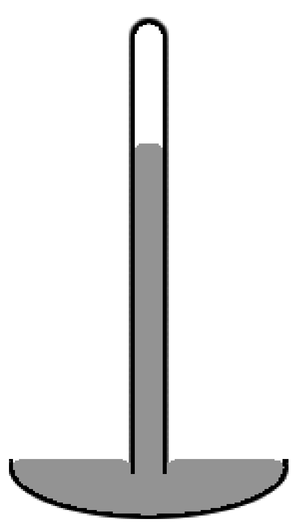

Fig. 4.4: Diagram of a Mercury Barometer

4.4. Cohesive and Adhesive Forces

There are several other important concepts that are related to surface tension. The first of these is the idea of cohesive and adhesive forces. Cohesive forces are those that hold the body of a liquid together with minimum surface area and adhesive forces are those that try to make a body of a liquid spread out. So if the cohesive forces are stronger than the adhesive forces, the body of water will maintain its shape, but if the opposite is true than the liquid will be spread out, *maximizing* its surface area. Any substance that you can add to a liquid that allows a liquid to increase its surface area is called a wetting agent.

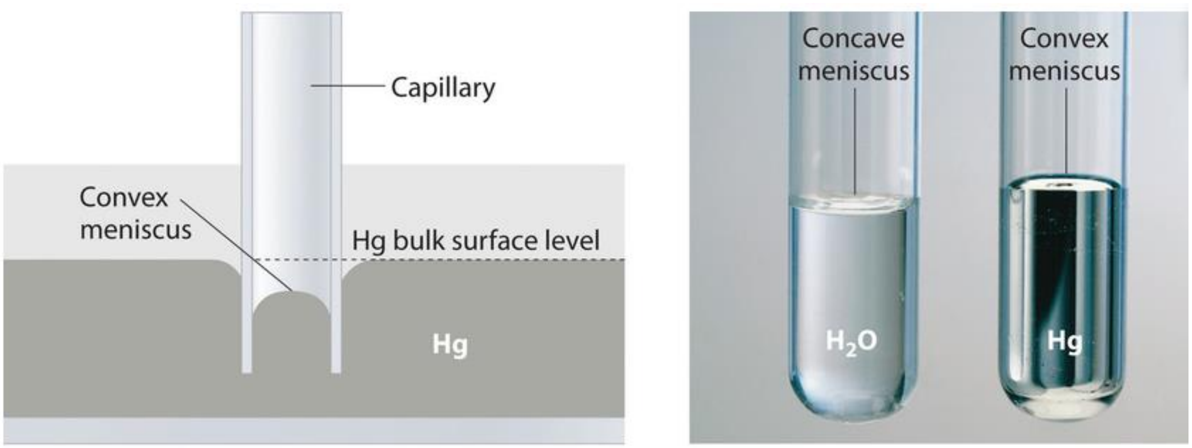

Fig. 4.5: Meniscus Shape

In the lab there are also several important points to remember about surface tension. The first you've probably noticed before. This is the idea of a meniscus . This is the concave (curved in) or convex (curved out) look that water or other liquids have when they are in test tubes. This is caused by the attraction between the glass and the liquid. With water, this causes it to climb up the sides of a test tube. This attraction is amplified as the diameter of the tubes increases; this is called capillary action. This can be seen if you take a tube with a very small diameter (a *capillary tube*) and lower it into a body of water. The liquid will climb up into the tube, even though there is no outside force. You may have also seen this when you put a straw into a drink and notice that the liquid level inside the straw is higher than it is in your drink. All of this however, requires that the adhesive forces (between the liquid and the capillary surface) be higher than the cohesive forces (between the liquid and itself), otherwise there will be no capillary action or the opposite can even happen. Mercury has higher cohesive forces than adhesive forces, so the level of the liquid will actually be lower in the capillary tubes than compared to the rest of the mercury.

4.4.1. Cohesion and Adhesion in Liquids

Children blow soap bubbles and play in the spray of a sprinkler on a hot summer day. An underwater spider keeps his air supply in a shiny bubble he carries wrapped around him. A technician draws blood into a small-diameter tube just by touching it to a drop on a pricked finger. A premature infant struggles to inflate her lungs. What is the common thread? All these activities are dominated by the attractive forces between atoms and molecules in liquids both within a liquid and between the liquid and its surroundings.

Attractive forces between molecules of the same type are called **cohesive forces**. Liquids can, for example, be held in open containers because cohesive forces hold the molecules together. Attractive forces between molecules of different types are called **adhesive forces**. Such forces cause liquid drops to cling to window panes, for example. In this section we examine effects directly attributable to cohesive and adhesive forces in liquids.

Cohesive forces between molecules cause the surface of a liquid to contract to the smallest possible surface area. This general effect is called **surface tension**. Molecules on the surface are pulled inward by cohesive forces, reducing the surface area. Molecules inside the liquid experience zero net force, since they have neighbors on all sides.

Surface tension is proportional to the strength of the cohesive force, which varies with the type of liquid. Surface tension$\gamma$is defined to be the force *F* per unit length L exerted by a stretched liquid membrane:

$$\gamma = \frac{F}{L}.$$

Surface tension is the reason why liquids form bubbles and droplets. The inward surface tension force causes bubbles to be approximately spherical and raises the pressure of the gas trapped inside relative to atmospheric pressure outside. It can be shown that the gauge pressure Pinside a spherical bubble is given by

$$P = \frac{4\gamma}{r}$$

Where r is radius of bubble.

Liquid	Surface Tensoin (N/m)
Water	0.0756
Water	0.0728
Water	0.0589
Soapy Water	0.0370
Ethyl alchohal	0.0223
Glycerin	0.0631
Mercury	0.435
Olive Oil	0.032
Tissue Fluid	0.050
Gold	0.058
Oxygen	0.0756
Blood Whole	0.0756
Blood Plasma	0.0756

4.5. Surface Energy Formula

Surface energy is the energy that exists between the surface molecules of solid materials or substances when a comparable attractive force exists. Low to high energies, or vice versa, are different. It is impossible to measure surface energy. It occurs as a result of a molecule-molecule interaction. A stretched membrane refers to the free surface of a liquid. The surface, which is known as Surface Free Energy, holds some Potential Energy on the liquid surface. Surface energy will be discussed in detail in this article.

The work done on the outer portion of a material when the atoms are not bonded to another atom in their immediate vicinity is known as surface energy. Any material's atoms must be joined to other atoms to function properly. This is due to the fact that connected atoms completely encircle the material's physical aspect, which remains constant. When the material reaches the surface, however, the atom's bonds rip open, and there are no bonds on the substance's outer surface. Surface energy is the technical term for this. Higher surface energy indicates that atoms are more motivated to rejoin links.

When a spring is stretched, some work is done on it, and the work is stored as Potential energy, also known as Elastic potential energy. There will be no potential energy if the body is in its undamaged state. When we talk about free surfaces of liquids, we know that they are stretched membranes, thus the surface will store some potential energy due to the stretched surface, which is referred to as Surface energy or Surface free energy because it is just at the liquid's surface.

Joules/m^2 or **Newton/meter (N/m)** is the SI unit for surface energy.

4.5.1. Formula of Surface Energy

The Surface Energy is calculated using the following formula:

Surface Energy = Work Done / Area

$E = S \times \Delta A$

Where,

- E = Surface Energy,
- S = Surface Tension,
- ΔA = Increase in Surface Area.

- **Surface Energy Dimensional Formula**
- The surface energy dimensional formula is as follows: **$[M^1L^0T^{-2}]$.**

(i) **Work Done in Blowing a Liquid Drop:** If a liquid drop is blown up from a radius r_1 to r_2, then work done for that is

$W = S \cdot 4\pi(r_2^2 - r_1^2)$

(ii) **Work Done in Blowing a Soap Bubble:** As a soap bubble has two free surfaces, hence work done in blowing a soap bubble so as to increase its radius from r_1 to r_2 is given by

$W = S \cdot 8\pi(r_2^2 - r_1^2)$

(iii) **Work Done in Splitting a Bigger Drop into n Smaller Droplets:** If a liquid drop of radius R is split up into n smaller droplets, all of same size, then radius of each droplet

$r = R\,(n)^{-1/3}$

Work done, $W = 4\pi S\,(nr^2 - R^2) = 4\pi SR^2\,(n^{1/3} - 1)$

(iv) **Coalescence of Liquid Drops:** If n small liquid drops of radius r each combine together so as to form a single bigger drop of radius $R = n^{1/3}\,r$, then in the process energy is released. Release of energy is given by

$\Delta U = S\,4\pi(nr^2 - R^2) = 4\pi Sr^2n\,(1 - n^{-1/3})$

4.6 Summery:

In this unit you have studied about surface tension. The attraction caused by the particles of any liquid at their surface that tends to resist the change is called as Surface Tension. It is the force applied on the surface of liquid per unit of its length.

4.7 Glossary:

Minimal: minimum

4.8 Solved Problems

Question 1: If the Surface Tension of water is 24×10^{-3} N/m and the Increase in Surface Area is 20 m. Find its Surface energy.

Solution:

$S = 24 \times 10^{-3}$ N/m, $\Delta A = 20$ m

Since,

$E = S \times \Delta A$

$\therefore E = 24 \times 10^{-3} \times 20$

$\therefore$ **E = 0.480 Joules/m^2**

Question 2: Find the Surface Tension if the Surface Energy is 32×10^{-3} Joules/m^2 and the Surface Area Increase is 12 m.

Solution:

$E = 32 \times 10^{-3}$ Joules/m^2, $\Delta A = 12$ m

Since,

$E = S \times \Delta A$

$\therefore S = E / \Delta A$

$\therefore S = 32 \times 10^{-3} / 12$

$\therefore$ **S = 2.666 × 10^{-3} N/m**

Question 3: If the liquid's surface tension is 40×10^{-3} N/m and the increase in surface area is 12 mm, Find out what its surface energy is.

Solution:

$S = 40 \times 10^{-3}$ N/m, $\Delta A = 12$ mm $= 12 \times 10^{-3}$ m

Since,

$E = S \times \Delta A$

∴ $E = 40 \times 10^{-3} \times 12 \times 10^{-3}$

∴ $E = 0.480 \times 10^{-3}$ Joules/m^2

Question 4: Assume that the Surface Tension is 9×10^{-3} N/m and the Increase in the Surface Area is 23 m then Find its Surface energy.

Solution:

$S = 9 \times 10^{-3}$ N/m, $\Delta A = 23$ m

Since,

$E = S \times \Delta A$

∴ $E = 9 \times 10^{-3} \times 23$

∴ $E = 0.207$ Joules/m^2

Question 5: Find Surface Energy when surface tension is 12×10^{-3} N/m and the Increase in Surface Area is 31 m.

Solution:

$S = 12 \times 10^{-3}$ N/m, $\Delta A = 31$ m

Since,

E

$= S \times \Delta A$

∴ $E = 12 \times 10^{-3} \times 31$

∴ $E = 0.372$ Joules/m^2

Question 6: Surface tension of water is 20×10^{-3} N/m and surface energy is 0.121 Joules/m^2 then find the increase in surface area.

Solution:

$S = 20 \times 10^{-3}$ N/m, $E = 0.121$ Joules/m^2 $= 121 \times 10^{-3}$ Joules/m^2

Since,

$E = S \times \Delta A$

$\therefore \Delta A = E / S$

$\therefore \Delta A = 121 \times 10^{-3} / 20 \times 10^{-3}$

$\therefore$ **$\Delta A = 6.05$ m**

4.9. SAQ (Self-Assessment Questions)

SAQ 1: The density of oil is less than that of water, yet a loaded oil tanker sits lower in the water than an empty one. Why?

SAQ 2: Is surface tension due to cohesive or adhesive forces, or both?

SAQ 3: Is capillary action due to cohesive or adhesive forces, or both?

SAQ 4: Birds such as ducks, geese, and swans have greater densities than water, yet they are able to sit on its surface. Explain this ability, noting that water does not wet their feathers and that they cannot sit on soapy water.

SAQ 5: Water beads up on an oily sunbather, but not on her neighbor, whose skin is not oiled. Explain in terms of cohesive and adhesive forces.

SAQ 6: Could capillary action be used to move fluids in a "weightless" environment, such as in an orbiting space probe?

SAQ 7: What effect does capillary action have on the reading of a manometer with uniform diameter? Explain your answer.

SAQ 8: Pressure between the inside chest wall and the outside of the lungs normally remains negative. Explain how pressure inside the lungs can become positive (to cause exhalation) without muscle action.

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UNIT 5 SIMPLE HARMONIC MOTION

Structure

5.1 INTRODUCTION

A periodic motion or harmonic motion is that repeats itself after a regular interval. The time interval after which the motion is repeated is called its time period. Some examples of periodic motion are motion of planets around the sun, motion of a piston inside a cylinder, used in automobile engines, motion of a ball in a bowl etc. as shown in figure 1.

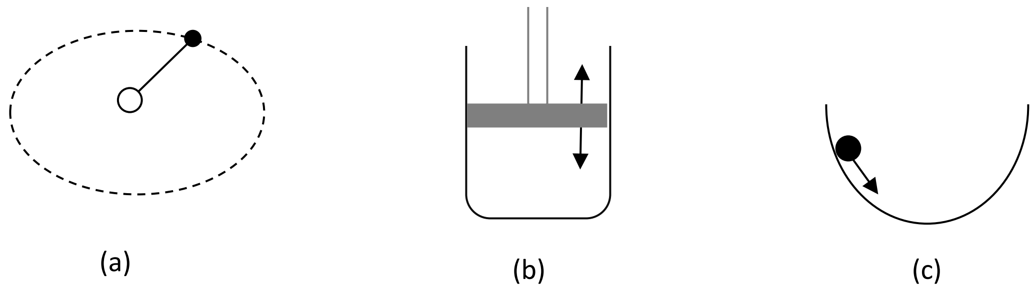


Figure 1: Some examples of periodic motion: (a) motion of the earth around the sun, or moon around the earth; (b) motion of a piston in a cylinder which is used in automobile engines; (c) motion of a ball in a bowl.

If in case of periodic motion, the body moves back and forth repeatedly about a fixed position (called equilibrium or mean position), the motion is said to be *oscillatory or vibratory*. For instance, the motion of the earth around the sun or the motion of the hands of the clock, are examples of periodic motion, but they are not oscillatory in nature. The motion of piston in an automobile engine, motion of a ball in a bowl, motion of needle of sewing machine or the bob of a pendulum clock are all examples of oscillatory motion. An oscillating body is said to execute *simple harmonic motion* (SHM) if the magnitude of the forces acting on it is directly proportional to the magnitude of its displacement from the mean position and the force (called restoring force) is always directed towards the mean position. Thus, SHM is actually a special case of oscillatory or vibratory motion. We will study SHM in detail in this unit. Some examples of simple harmonic motion include (see Fig. 2) are motion of a simple pendulum, a vibrating tuning fork, or a spring-mass system.

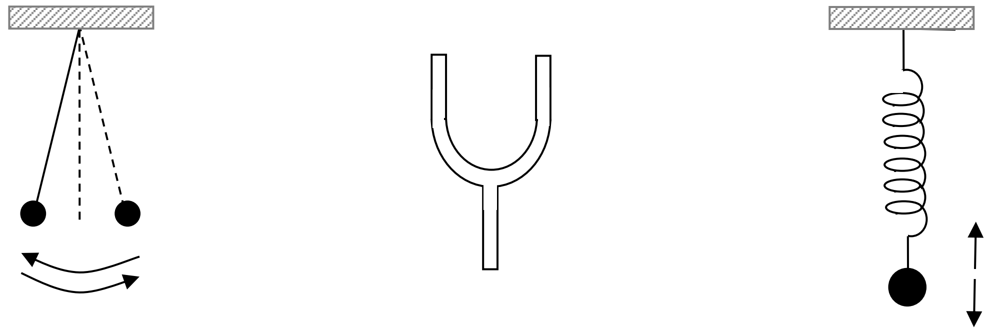


Figure 2: Some examples of SHM: (a) A simple pendulum; (b) a vibrating tuning fork; (c) an oscillating spring-mass system.

5.2 OBJECTIVES

After studying this unit, you should be able to

- understand simple harmonic motion
- understand the amplitude and the time period of an oscillating system
- write down the general equation of simple harmonic motion and solve it
- describe how the acceleration, velocity, angular frequency and displacement of an oscillating system change with time
- Damped harmonic motion
- Forced harmonic motion

5.3 OSCILLATORY MOTION

Any oscillating system moves to and fro (back and forth) repeatedly. Oscillations may be very complex such as those of a piano string or those of the earth during an earthquake or beating of the heart. There are also oscillations which are not very evident to our senses like the oscillations of the air molecules that transmit the sensation of sound, the oscillations of the atoms in a solid that convey the sensation of temperature or the oscillations of the electrons in the antennas of radio and TV transmitters. It would not be an exaggeration to say that we are indeed surrounded by oscillations all the time because oscillations are not just confined to material objects such as musical instruments but visible light, micro waves, radio waves and X-rays are also the outcome of oscillatory phenomena. Thus, the study of oscillations is essential for the understanding of various systems, be it mechanical, acoustical, electrical or atomic.

The oscillatory motion in a physical system results from two properties – the property of *inertia*, and the property of *elasticity*. We will begin with two illustrative physical systems which are described in the following sections. Studying such simple systems will help us in understanding the motion of more complicated oscillating systems.

5.3.1 Simple Pendulum

Do you remember that in your senior secondary class, you performed an experiment with a simple pendulum in your physics laboratory, where you measured the change in time period with the length of the string?

A simple pendulum consists of a heavy point mass, suspended from a fixed support through a weightless inextensible string. Here, we must understand that a simple pendulum is an idealized model. In practice, a simple pendulum is realized by suspending a small metallic sphere by a thread hanging from a fixed support like a stand. Fig. 3 shows a simple pendulum in which a bob of mass *m* is suspended from the fixed support *P* through a light string of length *l*. Left to itself, the bob occupies the position *PO*, with the angle $\theta = 0$, which is known as the mean or equilibrium position. From this mean position, the pendulum is drawn towards one extreme A such that the angle θ remains small. In doing so, the bob gains some finite potential energy. When the bob is released from *A*, it begins to move downward towards the mean position *O*. As a result, its potential energy begins to decrease as the bob approaches *O*. As the potential energy decreases, the bob gains kinetic energy. At the mean position, the bob's kinetic energy is maximum and its potential energy is minimum. Further. Due to having gained kinetic energy, the bob does not stop at the mean position; it overshoots the mean position and reaches the other extreme *B*. At position B, bob's potential energy becomes maximum and the kinetic energy is zero because its velocity becomes zero momentarily. After a momentary rest, the bob once again retraces its path from *B* to *O* to *A*. Thus, we see that the bob oscillates in a circular arc with the center at the point of suspension *P*.

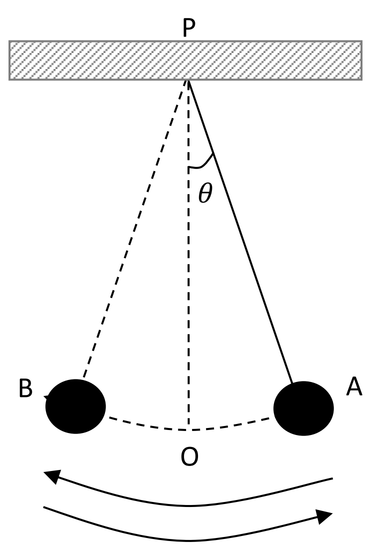


Figure 3: A simple pendulum.

Under ideal conditions, if there is no air resistance, losses due to friction do not affect the oscillatory motion. In such a situation, the pendulum should, in principle, oscillate forever! Each complete cycle of its oscillating motion takes it from one side of equilibrium to the other side and then back again.

5.3.2 Spring-Mass System

Just like simple pendulum, you must also be familiar with the spring pendulum.

Spring mass-system or spring pendulum consists of a weightless spring of constant *k*, one end of which is fixed rigidly to a wall and the other end is attached to a body of mass *m*, which is free to move horizontally or vertically depending on the system. If it is a loaded spring, it can move to and fro vertically. In the case of horizontal spring-mass system, the body is free to move on a frictionless horizontal surface, as shown in Fig. 4. When the spring is stretched, the elasticity of the spring tries to bring back the mass to its mean position. As the mass reaches the mean position, it has attained some velocity. As a result, the mass continues to move in the same direction and eventually compresses the spring until it reaches the other extreme position. The compressed spring pushes the mass back towards its mean position and the mass retraces its path. Thus, each cycle of oscillation takes the mass *m* from one extreme position to the extreme position on the other side of the mean position.

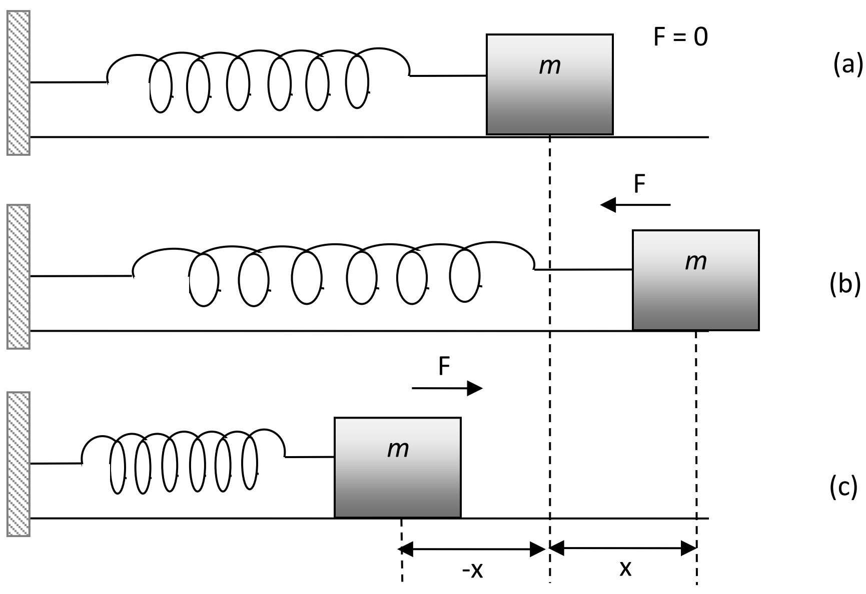


Figure 4: (a) Normal, (b) stretched, (c) compressed configurations of a horizontal spring-mass system.

Under ideal conditions, that is, if there is no air resistance and if the horizontal surface on which the mass is moving is frictionless, the spring – mass system should oscillate forever!

We will use spring-mass system, described above, to discuss the characteristics of SHM in the next section. You will also learn to calculate the force F shown in Fig. 4. But, before you proceed further, you should try to answer some questions based on what you have studied until now.

Self Assessment Question (SAQ) 1: In practice, the oscillations in a simple harmonic motion or a spring-mass system die away gradually and the mass *m* stops moving. What do you think is the reason for that?

Self Assessment Question (SAQ) 2: What are the two properties that are responsible for the oscillations?

Self Assessment Question (SAQ) 3: Do you think that the minute hand of the clock moves periodically? If so, can we also infer that its motion is oscillatory? Explain.

Self Assessment Question (SAQ) 4: Choose the correct option:

The motion of Halley's Comet around the sun is

(a) Periodic (b) Oscillatory (c) Simple harmonic (d) Translatory.

(Answer of Selected Self Assessment Questions (SAQs): 4. (a))

5.4 SIMPLE HARMONIC MOTION

If the periodic motion is such that the acceleration is of particle is always directly proportional to its displacement from its equilibrium position and always directed to equilibrium position (with negative direction of displacement), the motion of particle is said to be SHM.

$$a \propto (-x) \tag{1.1}$$

We can also define it as if the force acting on the oscillating body is always in the direction opposite to the displacement of the body from the equilibrium or and its magnitude is proportional to the magnitude of displacement, the body is said to be executing SHM.

$$F \propto (-x) \tag{1.2}$$

Similarly, for the spring-mass system, Hooke's Law states that the restoring force is proportional to the displacement of the spring in case of stretched as well as compressed configurations. In our case, the restoring force exerted by the spring on the body is directed to the left [see Fig. 4 (b)] and is given by the following relation:

$$F = -kx \tag{1.3}$$

Since, the restoring force, F is proportional to the displacement[1] and is opposite in sign to the displacement, the resulting motion is simple harmonic. Here k is called the spring constant or stiffness constant. The SI unit of k is Nm^{-1}.

Example 1: If, in a spring-mass system as shown in Fig. 4, the spring constant is 50 Nm^{-1} and the block of mass 1 kg is displaced by 0.01 m to the right before being released, calculate the

(a) restoring force at t = 0,
(b) restoring force when the block travels to the other extreme, and
(c) The restoring force in the static equilibrium position.

Solution:

(a) If x is taken as positive to the right of the mean position, then the restoring force is given by

$$F = -kx = -(50\ Nm^{-1})(0.01\ \text{m}) = -0.5\ N$$

(b) Similarly, the restoring force is given by

$$F = -kx = -(50\ Nm^{-1})(-0.01\ \text{m}) = +0.5\ N$$

(c) At the mean position, $x = 0$

$$F = -(50\ Nm^{-1})(0) = 0$$

5.4.1 Basic Characteristics of SHM

Since we now know what SHM is, let us define some of the basic characteristics of SHM. What comes to your mind? The first important characteristic in SHM is the initial displacement that actually results in oscillations in the first place. The magnitude of the initial displacement, which is also the maximum displacement, is called the ***amplitude (A)*** of oscillations. As we mentioned before, the energy of the system executing SHM alternates between kinetic and potential forms. At the extremities of the oscillations, the kinetic energy is zero as the velocity is zero and the potential energy is the maximum.

Another characteristic of SHM is the ***time period (T)*** which is the time taken for one complete cycle of oscillation. This is the least time taken by an oscillating object to move from a certain position and velocity back to the same position and velocity. Generally, for convenience, we measure the time period from either the mean position or the extreme ends.

Instead of time period, many a times we talk in terms of the ***frequency (ν)*** to characterize SHM. Frequency is the number of complete oscillations executed per second and is the inverse of the time period, i.e.

$$\nu = \frac{1}{T} \tag{1.4}$$

It is expressed in cycles per second or simply s^{-1} or hertz (Hz). We also define a term called ***angular frequency***, denoted by $\boldsymbol{\omega}$, which is given by

$$\omega = 2\pi\nu \qquad (1.5)$$

It is expressed in radian per second or simply$rad\ s^{-1}$, since 2π is the angle around a circle in radians and T is in seconds.

Example 2: A mass on a spring oscillates along a vertical line, taking 12 s to complete 10 oscillations. Calculate the time period, and the angular frequency.

Solution:

(a) Time period is the time taken for one complete cycle of oscillation; therefore, to complete one oscillation, time needed will be

$$T = \frac{(12\ s)}{(10\ oscillations)} = 1.2\ s$$

(b) The frequency is given by

$$\nu = \frac{1}{T} = \frac{1}{1.2}\ Hz$$

Therefore, the angular frequency is

$$\omega = 2\pi\nu = \frac{2\pi}{1.2} = 5.23\ rad\ s^{-1}$$

Self Assessment Question (SAQ) 5: An object executes simple harmonic motion with an angular frequency of $1.26 rad\ s^{-1}$. Calculate its time period.

Self Assessment Question (SAQ) 6: If the angular frequency ω is one revolution per minute. Calculate its time period. [Hint: One revolution = (2π) radians]

(Answer of selected SAQ 6. $\omega = 2\pi/60\ \ rad/s$ and $T = 1/30\ s$)

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{m(\frac{1}{k_1} + \frac{1}{k_2})}$$

5.5 DIFFERENTIAL EQUATION OF SHM

Let us now express equation (1.3) in the differential form by using Newton's second law of motion. From Newton's second law of motion, we know that force experienced by a body of mass m can be expressed as a function of acceleration,

$$F = ma = m\ddot{x} = m\frac{d^2x}{dt^2}$$

(double dot notation $\ddot{x} = \frac{d^2x}{dt^2}$)

Therefore, in a spring-mass system, the force can be written as

$$F = m\frac{d^2x}{dt^2} = -kx$$

Or we can say that

$$m\frac{d^2x}{dt^2} + kx = 0$$

$$or, \qquad \frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \qquad (1.6)$$

The above equation is the differential equation of SHM. k is the force constant (for our case of spring-mass system, it is called the spring constant) and has dimensions $(MLT^{-2}/L) = MT^{-2}$. Therefore, the dimension of k/m is T^{-2}, i.e. square of reciprocal of time. We can replace k/m by ω^2. Thus, the equation (1.6) takes the form

$$\ddot{x} + \omega^2 x = 0 \qquad (1.7)$$

We will find the physical meaning of ω, that it is actually the angular frequency that we already defined earlier, when we solve the differential equation (1.7).

5.5.1 Solution of the Differential Equation of SHM

The second time derivative of displacement ($\ddot{x}$) can be written as

$$\ddot{x} = \frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right)$$

Multiplying and dividing by dx in the numerator and the denominator, we get

$$\ddot{x} = \frac{dx}{dt}\frac{d}{dx}\left(\frac{dx}{dt}\right)$$

We already know that $\dot{x}$ or dx/dt actually define the velocity$v$. Therefore, the above expression can take the following form

$$\ddot{x} = v\frac{d}{dx}(v)$$

Since,

$$\frac{d}{dx}\left(\frac{v^2}{2}\right) = v\frac{dv}{dx}$$

We get

$$\ddot{x} = \frac{d}{dx}\left(\frac{v^2}{2}\right) \tag{1.8}$$

From (1.7) and (1.8), we get

$$\frac{d}{dx}\left(\frac{v^2}{2}\right) + \omega^2 x = 0$$

$$or \ \ \frac{d}{dx}\left(\frac{v^2}{2} + \omega^2\frac{x^2}{2}\right) = 0$$

$$\therefore d(v^2 + \omega^2 x^2) = 0 \tag{1.9}$$

On integrating both the sides, we get

$$v^2 + \omega^2 x^2 = constant\ (C_1) \tag{1.10}$$

We already know that on the two extremes, when the magnitude of the displacement is equal to the amplitude ($x = \pm A$), the kinetic energy or the velocity is zero ($v = 0$). Using this boundary condition in equation (1.10), we can calculate the constant (C_1). Thus, C_1 is given by

$$(0)^2 + \omega^2(\pm A)^2 = C_1$$

$$or \ \ C_1 = \omega^2 A^2$$

Using this value in equation (1.10) and rearranging the terms, we get

$$v^2 = \omega^2(A^2 - x^2)$$

$$or \ \ v = \pm\omega\sqrt{(A^2 - x^2)} \tag{1.11}$$

The above relation is the expression for velocity of a particle executing SHM. We can see how the velocity has a maximum magnitude at $x = 0$ or in other words, the mean position. From (1.11), the maximum velocity is given by

$$|v|_{max} = \omega A \tag{1.12}$$

Now, we will determine the expression for the displacement of a particle executing SHM. From (1.11), we get

$$\frac{dx}{dt} = \pm\omega\sqrt{(A^2 - x^2)}$$

Rearranging the terms, we get

$$\pm\frac{dx}{\sqrt{(A^2 - x^2)}} = \omega dt$$

On integrating both the sides, we get corresponding to the $(+)$ sign

$$\sin^{-1}\frac{x}{A} = \omega t + \delta_1$$

And, corresponding to the $(-)$ sign

$$\cos^{-1}\frac{x}{A} = \omega t + \delta_2$$

where δ_1 and δ_2 are dimensionless constants.

Therefore, we can see that the SHM is defined by a sinusoidal curve

$$\boldsymbol{x(t) = A\sin(\omega t + \delta)} \tag{1.13}$$

Depending on the value of constant δ and ωt the displacement from the equilibrium position and velocity of the SHM at any instant can be determined.

Example 4: A 50 g mass vibrates in SHM at the end of a spring. The amplitude of the motion is 12 cm and the period is 0.1 minutes. Find the maximum speed of the mass. What will be the speed at$x = A/2$?

Solution:

$$\omega = 2\pi\nu = 2\pi\left(\frac{1}{0.1\times 60\ s}\right) = 1.047\ rad\ s^{-1}$$

$$\therefore |v|_{max} = \omega A = (1.047\ rad\ s^{-1})(12\times 10^{-2}\ m)$$

$$= 0.1256\ m/s$$

From equation (1.11), we get

$$|v| = \omega\sqrt{A^2 - \left(\frac{A}{2}\right)^2} = \frac{3}{4}\omega A$$

$$= \frac{3}{4}(1.047\ rad\ s^{-1})(12\times 10^{-2}\ m) = 0.0942\ m/s$$

Self Assessment Question (SAQ) 7: In the above question, calculate the speed at $x = 1\ cm$.

Self Assessment Question (SAQ) 8: In the above question, at what location will the speed of the vibrating mass be 5 cm/s?

5.5.2 Angular Frequency of SHM

We know that the displacement $x(t)$ should return to its initial value after one time period T of the motion. Or

$$x(t) = x(t + T)$$

We also know from trigonometry that the sine or cosine function repeats itself when its argument has increased by $2\pi\ rad$. Thus,

$$\omega(t + T) = \omega t + 2\pi$$

Or, we get

$$\omega = \frac{2\pi}{T} = 2\pi\nu \tag{1.14}$$

The quantity ω is therefore, the angular frequency that we defined earlier. Its SI unit is $rad\ s^{-1}$.

From equation (1.6), we know that

$$\omega^2 = \frac{k}{m}$$

$$\therefore\ \omega = \sqrt{\frac{k}{m}} \tag{1.15}$$

Example 5: A particle of mass 0.2 kg undergoes SHM according to the equation: $x(t) = 3\sin(\pi t + \pi/4)$. [$t$ is in s and x in m]

(a) What is the amplitude of oscillation?
(b) What is the time period of oscillation?
(c) What is the initial value of x?
(d) What is the initial velocity when the SHM starts?
(e) At what instants is the particle's energy purely kinetic?

Solution:

(a) Comparing the given equation with $x(t) = A\sin(\omega t + \delta)$, we get the amplitude, $A = 3\ m$.

(b) On comparing, we get $\omega = \pi\ rad\ s^{-1}$. Therefore, from (1.14), we get the time period as

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\pi} = 2\ s$$

(c) Initial conditions are at $t = 0$

$$x(0) = 3\sin(\pi/4) = 1.5\sqrt{2}\ m$$

(d)

$$\frac{dx}{dt} = v(t) = 3\pi \sin(\pi t + \pi/4)$$
$$v(0) = 3\pi \sin(\pi/4) = \frac{3\pi}{\sqrt{2}}\ m/s$$

(e) The energy is purely kinetic when the particle is at the mean position, i.e. when $x(t) = 0$. Or

$$0 = 3\sin\left(\pi t + \frac{\pi}{4}\right)$$
$$\therefore \pi t + \frac{\pi}{4} = 0, \pi, 2\pi, 3\pi, \ldots$$
$$i.e.\ \ t = -\frac{1}{4}, \frac{3}{4}, \frac{7}{4}, \frac{11}{4}, \ldots$$

Rejecting the negative value of t, we get $t = 3/4, 7/4, 11/4\ldots$ At these instants, the particle crosses origin and hence its energy is purely kinetic.

Self Assessment Question (SAQ) 9: How are the following characteristics of SHM affected by doubling the amplitude? Explain.

(a) Time period, and (b) maximum velocity.

Self Assessment Question (SAQ) 10: Choose the correct option:

Which of the following functions represent SHM?

(a) $\sin(2\omega t)$ (b) $\sin^{-1}\omega t$ (c) $\sin(\omega t) + 2\cos(\omega t)$ (d) $\sin(\omega t) + \cos(2\omega t)$

(Answer SAQ 9. Period remains unchanged. Maximum velocity is doubled. SAQ 10. (a))

Example 6: A copper spring suspended from a fixed point supports a scale pan of mass 0.05 kg at equilibrium. The scale pan descends 40 mm to a new equilibrium position when a 1 N weight is placed on it. Calculate the

(a) spring constant,
(b) the total mass of the scale pan and the 1 N weight. [g can be taken as 10 ms^{-2}]
(c) The scale pan, with 1 N weight on it, is pulled a distance of 15 mm downwards from equilibrium and then released. Calculate the time period of the oscillations, and
(d) the maximum speed of the scale pan.

Solution:

(a) From equation (1.18),

$$k = \frac{F}{d} = \frac{1}{0.04} = 25\ N/m$$

(b) Mass of 1 N weight = weight/g = 0.1 kg. Therefore, the total mass = 0.1 + 0.05 = 0.15 kg.
(c) The time period is given as

$$T = 2\pi\sqrt{\frac{0.15}{25}} = 0.49\ s$$

(d) The amplitude of the oscillations = 15 mm = 0.015 m. Therefore,

$$|v|_{max} = \omega A = \left(\frac{2\pi}{0.49\ s}\right)(0.015\ m) = 0.195\ m/s$$

Self Assessment Question (SAQ) 11: A steel spring, suspended from a fixed point, supports a 0.2 kg stone hung from its lower end. The stone is displaced downwards from its equilibrium position by a distance of 25 mm and then released. The time for 20 oscillations is measured as 22 s. Calculate (a) its time period, (b) its angular frequency, (c) its maximum speed, (d) the maximum tension in the spring.

(Answer 11. Hint: The maximum tension in the spring will be when it is stretched to the extreme, which is equal to the sum of the difference of the relaxed length and the equilibrium length of the spring, and the amplitude of the oscillations; i.e. $d + A = (T_0 + \Delta T)/k$.)

5.6 SHM as Projection of Circular Motion

Consider a particle P moving on a circular path of radius r as shown in Fig. 2. We can write the coordinates of point P as $x = r\cos\theta$ and $y = r\sin\theta$, where r is the radius of the circle and θ is the angle between the line OP and the x-axis.

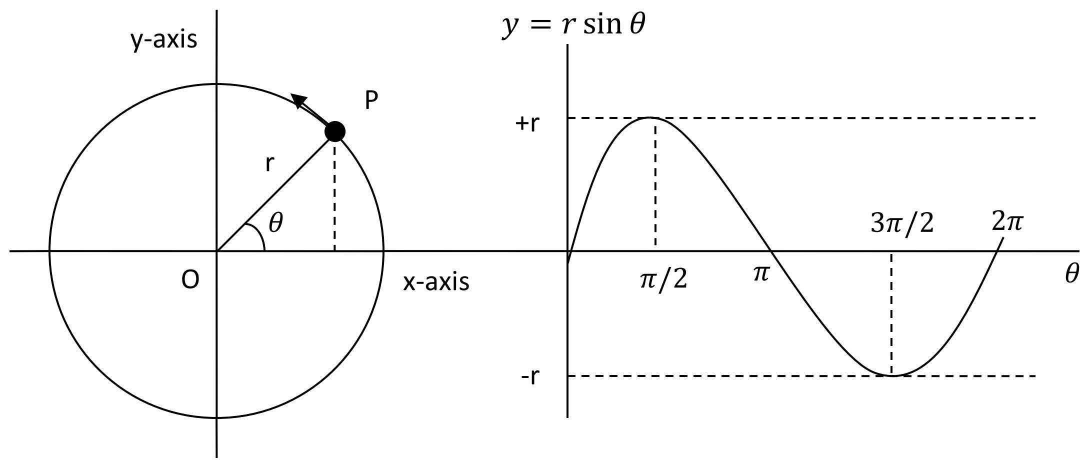


Figure 5: A particle P moving on a circular path. Its projection on the diameter generates a sinusoidal curve.

$x = r\cos\theta\, y = r\sin\theta\, r\theta$. From Fig. 2, you may note that, as the particle P moves along the circular path,, its y-coordinate ($= r\sin\theta$) changes because θ changes from 0 to 2π. Thus, we can see that the y-coordinate of the particle P executes SHM. Similarly, you should convince yourself that the x-coordinate ($= r\cos\theta$) of the particle P will also execute SHM. However, the phases of the two harmonic motions differ by $\pi/2$ as $\cos\theta = \sin(\theta + \pi/2)$.

Therefore, the projection of a uniform circular motion on a diameter of the circle is a simple harmonic motion. This representation **of SHM** is known as the **rotating vector representation**.

5.7 VELOCITY AND ACCELERATION IN SHM

From your school mathematics, you may recall that, if we know the expression for the displacement of a particle, we can obtain expressions for its velocity and acceleration using differential calculus. In the previous Section, we obtained the expression for the velocity of the particle executing SHM by differentiating the expression for displacement, x(t):

$$x(t) = A\sin(\omega t + \delta)$$

$$\frac{dx}{dt} = v(t) = A\omega\cos(\omega t + \delta)$$

From the above expression, you may note that the amplitude of velocity or maximum velocity is given by

$$|v|_{max} = \omega A$$

In previous section, we obtained an expression for the velocity in terms of displacement and other parameters of SHM:

$$v(x) = \pm\omega\sqrt{(A^2 - x^2)}$$

To show that the above two expressions for the velocity are equivalent, we write $x(t) = A\sin(\omega t + \delta)$:

$$v = \pm\omega\sqrt{A^2\{1 - \sin^2(\omega t + \delta)\}}$$

From basic trigonometry, we know that $1 - \sin^2\theta = \cos^2\theta$ is an identity. Therefore, we get $v = \pm A\omega\cos(\omega t + \delta)$.

Further, to obtain the expression for the acceleration of the particle executing SHM, we shall differentiate the expression for displacement twice, i.e.

$$\frac{d^2x(t)}{dt^2} = a(t)$$

$$or \quad \frac{d^2\{A\sin(\omega t + \delta)\}}{dt^2} = a(t)$$

$$\therefore a(t) = -A\omega^2\sin(\omega t + \delta) \tag{1.16}$$

The acceleration can also be expressed in terms of the displacement of the particle, i.e.

$$a(t) = -A\omega^2 x(t) \tag{1.17}$$

From equation (1.17), you may note that the acceleration in SHM is always directed towards the mean position. The magnitude of acceleration is minimum at the mean position and maximum at the extremes.

$$\begin{aligned} |a|_{min} &= 0 \quad && at\ mean\ position \\ |a|_{max} &= \omega^2 A \quad && at\ the\ two\ extremes \end{aligned} \tag{1.18}$$

Using the expression for acceleration, we can determine the restoring force acting on the oscillating object:

$$F(t) = ma(t) = -mA\omega^2\sin(\omega t + \delta) \tag{1.19}$$

At any position x, it is given by

$$F(x) = -m\omega^2 x \tag{1.20}$$

We also know that $F = -kx$ in case of spring-mass system. Comparing it with equation (1.20) gives us the familiar expression for angular velocity

$$\omega = \sqrt{\frac{k}{m}}$$

5.8 TRANSFORMATION OF ENERGIES IN SHM

While discussing the motion of simple oscillatory systems, we discovered that the energy of the oscillation alternates between potential and kinetic forms; the potential energy being minimum at the mean position and maximum at the extremities. On the other hand, the kinetic energy is maximum at the mean position and minimum at the extremities. While the sum of potential energy (U) and kinetic energy (K), which is the total mechanical energy (E) of the oscillator, remains constant. Let us now derive an expression for the potential, kinetic and total mechanical energy in SHM.

5.8.1 Potential Energy

We shall derive the elastic potential energy of the simple spring – mass system that we studied in Unit 1. The value of the elastic potential energy of the spring-mass system depends entirely on how much the spring is stretched or compressed, i.e. the displacement $x(t)$ of the mass from its equilibrium position$x(t)$Further, the elastic potential energy dU gained by the system is equal to the work done against the force in moving it through a distance dx. In other words,

$$dU = -F(x)dx \tag{1.21}$$

Replacing $F(x) = -m\omega^2 x$ in the above equation, we get

$$dU = m\omega^2 x\, dx$$

Thus, the total elastic potential energy at a point x will be equal to the total work done in moving the oscillator from the mean position ($x = 0$). Therefore, integrating the above expression from 0 to x, we get

$$U = m\omega^2 \int_0^x x\, dx$$

$$or \quad U = \frac{1}{2} m\omega^2 x^2$$

$$\therefore U = \frac{1}{2} kx^2 = \frac{1}{2} kA^2 \sin^2(\omega t + \delta) \tag{1.22}$$

Let us also calculate the average potential energy of the spring-mass system over one complete cycle. This can be determined by integrating it over time from 0 to T, i.e. one time period. Thus,

$$\langle U \rangle = \frac{1}{2}kA^2\left[\frac{\int_0^T \sin^2(\omega t+\delta)dt}{\int_0^T dt}\right]$$

$$= \frac{1}{2}kA^2\left[\frac{1}{2}\right]$$

$$\therefore \langle U \rangle = \frac{1}{4}kA^2 \tag{1.23}$$

5.8.2 Kinetic Energy

The kinetic energy of the spring-mass system is entirely associated with the moving object. Its value depends on how fast the object is moving, that is, on $v(t)$. Hence,

$$K = \frac{1}{2}mv^2$$

$$\therefore K = \frac{1}{2}m\omega^2A^2\cos^2(\omega t+\delta) = \frac{1}{2}kA^2\cos^2(\omega t+\delta) \tag{1.24}$$

Therefore, the average kinetic energy, which can be calculated by integrating it over time from 0 to T, i.e. one time period, will be

$$\langle K \rangle = \frac{1}{2}kA^2\left[\frac{\int_0^T \cos^2(\omega t+\delta)\,dt}{\int_0^T dt}\right]$$

$$= \frac{1}{2}kA^2\left[\frac{1}{2}\right]$$

$$\therefore \langle K \rangle = \frac{1}{4}kA^2 \tag{1.25}$$

Thus, we find that the average potential energy of the spring-mass system is equal to its average kinetic energy.

5.8.3 Total Mechanical Energy

Using equations (1.22) and (1.24), we can determine the total mechanical energy at a particular instant, by summing the potential and the kinetic energies,

$$E = U + K$$

$$= \frac{1}{2}kA^2\sin^2(\omega t+\delta) + \frac{1}{2}kA^2\cos^2(\omega t+\delta)$$

$$= \frac{1}{2}kA^2[\sin^2(\omega t+\delta) + \cos^2(\omega t+\delta)]$$

From trigonometry, we know that $\sin^2\theta + \cos^2\theta = 1$ is an identity. Thus,

$$E = \frac{1}{2}kA^2 \tag{1.26}$$

The total mechanical energy of the oscillator (spring-mass system) is indeed a constant and is independent of time or position.

The potential energy and kinetic energy of a linear oscillator are shown as the function of time in the figure below. Note that all the energies are positive and that the potential energy and the kinetic energy peak twice during every period.

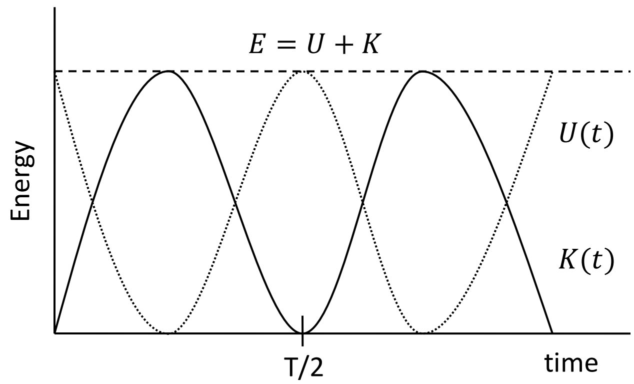


Figure 6: Potential energy, kinetic energy and total energy as functions of time, for SHM.

Next, in Fig. 5, we show the variation of potential energy and kinetic energy of a linear oscillator as the function of displacement. Note that, at $x = 0$, that is, at the mean position, the energy is all kinetic while at the extremities, i.e.at $x = \pm A$, it is all potential.

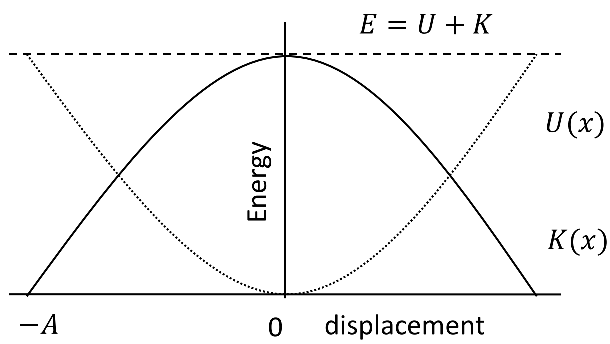


Figure 7: Potential energy, kinetic energy and total energy as functions of position, for SHM.

Self Assessment Question (SAQ) 12: Choose the correct option:

A body is in SHM. The motion is represented graphically. The valid representation of the position will be

(a) A square wave
(b) A straight line
(c) A sinusoidal curve
(d) A $(y = x^2)$ curve
(e) A curve of the form $y = 5|\sin \varphi|$

Self Assessment Question (SAQ) 13: In the previous question, what will be the valid representation of the velocity?

Self Assessment Question (SAQ) 14: In the previous question, what will be the valid representation of the acceleration?

Self Assessment Question (SAQ) 15: Choose the correct option:

For a particle executing SHM, which of the following statements does not hold good?

(b) The total energy of the particle always remains the same.
(c) The restoring force is always directed towards a fixed poin.t
(d) The restoring force is maximum at the extreme positions.
(e) The acceleration of the particle is minimum at the mean position.
(f) The velocity of the particle is minimum at the mean position.

(SAQ) 16: Two simple harmonic motions are represented by the equations $x_1 = 10\sin(3t + \pi/4)$ and $x_2 = 5\cos(9t + \pi/3)$. Their amplitudes are of the ratio ______________.

(SAQ) 17: Choose the correct option:

A body is in SHM. The motion is represented graphically. The valid representation of the position will be

(f) A square wave
(g) A straight line
(h) A sinusoidal curve
(i) A $(y = x^2)$ curve
(j) A curve of the form $y = 5|\sin \varphi|$

(SAQ) 18: In the previous question, what will be the valid representation of the velocity?

(SAQ) 19: In the previous question, what will be the valid representation of the acceleration?

(SAQ) 20: Choose the correct option:

For a particle executing SHM, which of the following statements does not hold good?

(g) The total energy of the particle always remains the same.
(h) The restoring force is always directed towards a fixed poin.t
(i) The restoring force is maximum at the extreme positions.
(j) The acceleration of the particle is minimum at the mean position.
(k) The velocity of the particle is minimum at the mean position.

(Answers 16. Ratio of amplitudes = 10:5 or 2:1, 17. (c) , 18. (c) , 19. (c), 20. (e))

Example 7: Show that the sine and the cosine functions describing the displacement of the oscillating body executing SHM are equivalent.

Solution: The general expression for displacement is given by

$$x(t) = A\sin(\omega t + \delta)$$

Defining another arbitrary constant δ_1 such that $(\pi/2 + \delta_1) = \delta$, the above expression may be written as

$$x(t) = A\sin(\omega t + \pi/2 + \delta_1) = A\cos(\omega t + \delta_1)$$

Therefore, we can say that the sine and the cosine forms are equivalent. The value of phase constant, however, depends on the form chosen.

Example 8: A particle starts at $t = 0$ from the mean position with a velocity $v = 3\pi\ m/s$ in the positive direction. If the time period of the oscillation is 2 sec., write the expression for the displacement of the particle.

(a) What minimum time does the particle take to go from mean position to a point P, which lies midway between the mean position and the right extreme position?
(b) What minimum time does the particle take to reach the right extreme position from the mean position?

Solution: From equation (1.14), we know that

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2\ s} = \pi\ rad\ s^{-1}$$

Let $x(t) = A\sin(\omega t + \delta)$ be the expression for the displacement of the particle executing SHMon. Therefore, the velocity is given by

$$\frac{dx}{dt} = v(t) = A\omega\cos(\omega t + \delta)$$

Applying the initial conditions, at $t = 0$

$$x(0) = 0\ m;\ \ v(0) = 3\pi\ m/s$$

On the above expressions for displacement and velocity, we get

$$0 = A\sin(\delta)$$

$$\therefore \delta = 0, \pi$$

And

$$3\pi = A\omega\cos(\delta)$$

Hence, $\delta = 0$ is possible but $\delta = \pi$ is not possible. Therefore, $\delta = 0$ is a possible solution. Substituting it in the above equation, we get

$$3\pi = A\omega\cos(0) = A\omega$$

$$A = \frac{3\pi}{(\pi\ rad\ s^{-1})} = 3\ m$$

Therefore, the equation of motion is $x(t) = 3\sin(\pi t)$

(a) The particle is at the mean position at $t = 0$. Let us assume that the particle reaches the point P (midway between the mean position and the right extreme) from its mean position in time t. Thus, we have, $x(t) = A/2\ = 1.5\ m$.
Thus, $1.5 = 3\sin(\pi t)$ or t = 1/6 s.

(b) $x(t) = A\ = 3\ m$. Thus, $\sin(\pi t) = 1$ or t = 0.5 s.

Example 9: A block, whose mass is *680 g*, is fastened to a spring whose spring constant *k* is *65 N/m*. The block is pulled a distance *x = 11 cm* from its equilibrium position at *x = 0* on a frictionless horizontal surface and released from rest at *t = 0*.

(a) What force does the spring exert on the block just before the block is released?
(b) What are the angular frequency, the frequency, and the period of the resulting oscillation?
(c) What is the amplitude of the oscillation?
(d) What is the maximum speed of the oscillating block?
(e) What is the magnitude of the maximum acceleration of the block?
(f) What is the phase angle for the motion?
(g) What is the total mechanical energy of the oscillator?
(h) What is the potential energy of this oscillator when the block is halfway to its end-point?
(i) What is the kinetic energy of the oscillator when the block is halfway to its end-point?

Solution:

(a) From Hooke's law

$$F = -kx = -(65)(0.11) = -7.2\ N$$

(b) For the givem spring-mass system, the angular frequency is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{65}{0.68}} = 9.78\, rad/s$$

Thus, the frequency is

$$\nu = \frac{\omega}{2\pi} = \frac{9.78}{2\pi} = 1.56\, Hz$$

And the time period is

$$T = \frac{1}{\nu} = \frac{1}{1.56} = 0.64\, s$$

(c) Since the block is released from rest at 11 cm distance from its equilibrium point, the kinetic energy it possesses at this point is zero. We already know that at the position of maximum displacement, the energy is all potential and the kinetic energy is zero. Hence, the amplitude A should be equal to 11 cm or 0.11 m.

(d) The maximum speed is given by

$$|v|_{max} = \omega A = (9.78)(0.11) = 1.1\, m/s$$

(e) The maximum acceleration is when the block is at the ends of its path. At those points the force acting on the block has its maximum magnitude.

$$|a|_{max} = \omega^2 A = (9.78)^2(0.11) = 11\, ms^{-2}$$

(f) At $t = 0$, when the block is released, the displacement of the block has maximum value equal to the amplitude and the velocity of the block is zero. Using these initial conditions, we get

$$1 = \sin\delta$$

And

$$0 = \cos\delta$$

The smallest angle that satisfies both these conditions is $\delta = \pi/2$.

Note: Any angle $(2n\pi + \pi/2)$ *rad, where n is an integer, will also satisfy these conditions.*

(g) We already know that the total energy will be constant.

$$E = \frac{1}{2}kA^2 = \frac{1}{2}(65)(0.11) = 0.393\, J$$

(**Comment**: correct the above equation: it should be $(0.11)^2$)

(h) The potential energy is given by

$$E = \frac{1}{2}kx^2 = \frac{1}{2}k\left(\frac{A}{2}\right)^2$$

$$= \frac{1}{8}kA^2 = \frac{1}{4}E = 0.098\, J$$

(**Comment**: check the arithmetic above; it will change in view of correction in (g) above)

(i) The kinetic energy can be determined by subtracting the potential energy component from the total energy

$$K = E - U$$
$$= 0.393 - 0.098 = 0.295\,J$$

Thus, we see at this position during the oscillation, about 25% of the energy is in the potential form and the rest 75% is in kinetic form.

5.9 DAMPED HARMONIC OSCILLATOR

Every physical system experiences damping, and damping depends upon the system under consideration. A familiar example is a spring – mass system executing longitudinal oscillations in a horizontal surface. The mass which has to move on the horizontal surface experiences frictional force from the surface and this frictional force opposes its motion. So, the friction due to the surface acts like damping force for the oscillating spring-mass system. In general, inclusion of damping force makes mathematical analysis somewhat difficult. But for simplicity, it is customary to model it by an equivalent viscous damping. In our discussion, we make the simplifying assumption that velocity of the moving part of the system is small so that the damping force can be taken to be linear in velocity.

For studying the effect of damping on a one dimensional oscillator, we can consider the representative case of a spring-mass system, as shown in figure below.

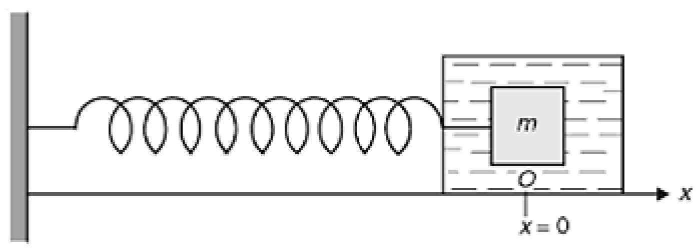


Figure 8: A damped spring-mass system

The spring-mass system in which the oscillating mass is executing oscillations in a viscous medium which causes its amplitude progressively decreasing to zero is called a *damped harmonic oscillator*. Obviously, in case of such an oscillator, in addition to the restoring force $-kx$, a resistive or damping force also acts upon it. This damping force is proportional to the velocity, v (= dx/dt). We, therefore, can write the ***differential equation of the damped harmonic oscillator*** as

$$m\frac{d^2x}{dt^2} = -\gamma\frac{dx}{dt} - kx$$

or $$\frac{d^2x}{dt^2}+\frac{\gamma}{m}\frac{dx}{dt}+\frac{k}{m}x=0 \qquad \text{-------------------------- (1.27)}$$

This can further be written as

$$\frac{d^2x}{dt^2}+2b\frac{dx}{dt}+\omega_0{}^2x=0 \qquad \text{------------------------------ (1.28)}$$

where $\frac{k}{m}=\omega_0{}^2$ is the natural frequency of oscillating particle (i.e. its frequency in the absence of damping), $\frac{\gamma}{m}=2b$ (k is the damping constant of the resistive medium)

Above equation is called the **differential equation of a damped harmonic oscillator**.

5.9.1 SOLUTION OF THE DIFFERENTIAL EQUATION OF DAMPED HARMONIC OSCILLATOR

The above differential equation is a second order linear homogeneous differential equation. Therefore, it will have at least one solution of type $x=Ae^{\alpha t}$

Here α and t both are arbitrary constants.

Therefore,

$$\frac{dx}{dt}=\alpha Ae^{\alpha t} \quad \text{and} \quad \frac{d^2x}{dt^2}=\alpha^2Ae^{\alpha t}$$

Substituting these values in the differential equation (6) above we get

$$\alpha^2Ae^{\alpha t}+2b\alpha Ae^{\alpha t}+\omega_0{}^2Ae^{\alpha t}=0$$

Or

$$\alpha^2+2b\alpha+\omega_0{}^2=0 \qquad \text{---------------------------- (1.29)}$$

This is a quadratic equation in α having its solution of the form

$$\alpha=-b\pm\sqrt{b^2-\omega_0^2}$$

Thus the original differential equation is satisfied by following two values of x

$$x=Ae^{(-b+\sqrt{b^2-\omega_0^2})t}$$

and $x = Ae^{(-b-\sqrt{b^2-\omega_0^2})t}$

Since the equation being a linear one, the linear sum of two linearly independent solutions will also be a general solution.

Therefore,

$$x = A_1 e^{(-b+\sqrt{b^2-\omega_0^2})t} + A_2 e^{(-b-\sqrt{b^2-\omega_0^2})t} \qquad \text{-----------------} (1.30)$$

Here A_1 and A_2 are arbitrary constants.

$$\text{Or} \quad x = A_1 e^{-\frac{t}{2\tau}+\beta t} + A_2 e^{-\frac{t}{2\tau}-\beta t} \qquad \text{----------------------} (1.31)$$

where $b = \dfrac{1}{2\tau}$ and $\beta = \sqrt{b^2 - \omega_0^2}$

The values of the constants A_1 and A_2 can be determined as given below:

Differentiating Eq. (1.31) with respect to t, we get

$$\frac{dx}{dt} = (-\frac{1}{2\tau} + \beta)A_1 e^{-\frac{t}{2\tau}+\beta t} + (-\frac{1}{2\tau} - \beta)A_2 e^{-\frac{t}{2\tau}-\beta t} \qquad \text{---------------} (1.32)$$

Now at t=0, displacement must be maximum, i. e. $x_{max}=a_0=A_1+A_2$ and $\dfrac{dx}{dt} = 0$

Putting t=0 in Eq. (1.32)

$$(-\frac{1}{2\tau} + \beta)A_1 + (-\frac{1}{2\tau} - \beta)A_2 = 0$$

$$-\frac{1}{2\tau}(A_1 + A_2) + \beta(A_1 - A_2) = 0$$

$$-\frac{1}{2\tau}(a_0) + \beta(A_1 - A_2) = 0$$

$$\beta(A_1 - A_2) = \frac{a_0}{2\tau}$$

$$\text{Or} \quad (A_1 - A_2) = \frac{a_0}{2\tau\beta} \qquad \text{----------------} (1.33)$$

As we know $A_1 + A_2 = a_0$

Adding it with (1.33), we get

$$A_1 = \frac{a_0}{2}\left[1 + \frac{1}{2\tau\beta}\right]$$

And $A_1 = (A_1 + A_2) - A_1$

$$= a_0 - \frac{a_0}{2}\left[1 + \frac{1}{2\tau\beta}\right]$$

$$= \frac{a_0}{2}\left[1 - \frac{1}{2\tau\beta}\right]$$

Putting these values in equation (1.31), we get-

$$x = \frac{a_0 e^{-\frac{t}{2\tau}}}{2}\left[\left(1 + \frac{1}{2\tau\beta}\right)e^{\beta t} + \left(1 - \frac{1}{2\tau\beta}\right)e^{-\beta t}\right] \quad \text{---- --------- (1.34)}$$

For analysis purpose, above equation may be written as

$$x = \frac{a_0 e^{-\frac{t}{2\tau}}}{2}\left[\left(1 + \frac{1}{2\tau\beta}\right)e^{(\sqrt{b^2-\omega_0^2})t} + \left(1 - \frac{1}{2\tau\beta}\right)e^{-(\sqrt{b^2-\omega_0^2})t}\right] \quad \text{--------- (1.35)}$$

Now Eq. (1.35) can be discussed according to following three cases.

CASE I: WHEN b (OR $\frac{1}{2\tau}$)>ω_0, CASE OF OVERDAMPING

In such case $\sqrt{(b^2 - \omega_0^2)}$ is a real quantity, with a positive value. This means that each term in the R. H. S. of Eq. (13), has an exponential term with a negative power. Therefore, the displacement of the oscillator, after attaining a maximum, dies off exponentially with time. Thus, after some time, there will be no oscillations. Such kind of oscillatory motion is called **overdamped** or **aperiodic** motion. Such kind of motion we see in case of dead beat galvanometer.

CASE II: WHEN b (OR $\frac{1}{2\tau}$)= ω_0, CASE OF CRITICAL DAMPING

In such case $\sqrt{(b^2 - \omega_0^2)} = 0$ .Therefore, each term on R. H. S. of Eq. (1.35 13) becomes infinite.

Still we can assume that, $\sqrt{(b^2 - \omega_0^2)} = h$ (where h is a very small quantity but not zero obviously).

Therefore Equation (8) gives-

$$x = A_1\, e^{(-b+h)t} + A_2\, e^{(-b-h)t}$$

$$= e^{-bt}\,(A_1\, e^{ht} + A_2\, e^{-ht})$$

$$= e^{-bt}\,[A_1(1 + ht + \frac{h^2t^2}{2!} + \frac{h^3t^3}{3!} + \ldots\ldots\ldots\ldots) + A_2(1 - ht + \frac{h^2t^2}{2!} - \frac{h^3t^3}{3!} + \ldots\ldots\ldots)]$$

Neglecting the terms containing higher powers of h, we obtain-

$$x = e^{-bt}\,[A_1(1 + ht) + A_2(1 - ht)]$$

$$= e^{-bt}\,[(A_1 + A_2) + (A_1 - A_2)ht]$$

$$= e^{-bt}\,[M + Nt] \quad \text{-------------------------------} \quad (1.36)$$

Here $(A_1 + A_2) = M$ and $(A_1 - A_2)h = N$

Further at $t = 0,\ x = x_{max} = a_0$

And $\frac{dx}{dt} = 0$

Therefore, the above equation becomes

$$a_0 = M$$

Differentiating Eq. (1.36), we get

$$\frac{dx}{dt} = \frac{d}{dt}(\mathrm{M\ e^{-bt}}) + \frac{d}{dt}(\mathrm{Nte^{-bt}})$$

$$0 = -\,b\,M\,e^{-kt} + Ne^{-bt} - Nte^{-bt}$$

$$= -bM + N$$

Or $N = ba_0$

Putting these values of M and N in equation (1.36) above

$$x = e^{-bt}\,(a_0 + ba_0t)$$

$$= a_0e^{-bt}\,(1 + bt)$$

$$= a_0\, e^{-\frac{t}{2\tau}}\,(1 + \frac{t}{2\tau})$$

$$= a_0 e^{-\frac{t}{2\tau}} + a_0 e^{-\frac{t}{2\tau}} \left(\frac{t}{2\tau}\right)$$

An important feature of the above expression is that its second term decays less rapidly as compared to its first term. In such cases, the displacement of the oscillator first increases, then quickly return back to its equlibrium position. This kind of oscillatory motion is known as *just aperiodic* (it just ceases to oscillate), or non oscillatory. This case is known as the **critical damping**.

Critical damping finds many applications in many pointer type instruments like, galvenometers, where the pointer moves to and stays at, the correct position, without any further oscillations.

CASE III: WHEN b (OR $\frac{1}{2\tau}$) < ω_0 ,CASE OF WEAK (UNDER) DAMPING

In such cases, the quantity $\sqrt{(b^2 - \omega_0^2)}$ will be imaginary one.

Let $\sqrt{(b^2 - \omega^2)} = i\omega$, where i = $\sqrt{(-1)}$ and $\omega = \sqrt{(\omega_0^2 - b^2)}$ is a real quantity

Putting the values –

$$x = A_1 e^{(-b+iw)t} + A_2 e^{(-b-iw)t}$$

$$= e^{-bt}[A_1(\cos \omega t + i \sin\omega t) + A_2(\cos \omega t - i\sin\omega t)]$$

$$= e^{-bt}[\cos \omega t\, (A_1 + A_2) + \sin \omega t\{i\,(A_1 - A_2)\}]$$

$$= e^{-bt}[A \cos \omega t + B \sin \omega t]$$

where $(A_1 + A_2) = A$ and $i(A_1 - A_2) = B$

$$= e^{-kt}[\, a_0 \cos \omega t \,.\, \frac{A}{a_0} + a_0 \sin \omega\, t \,.\, \frac{B}{a_0}]$$

Considering a right angle triangle as below in Fig. 3.

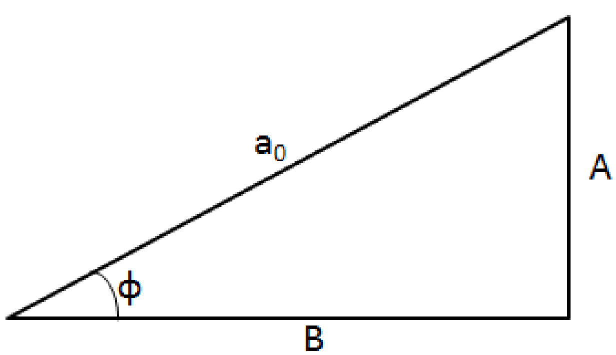


Figure 6

Therefore, we can write

$$sin\ \varphi = \frac{A}{a_0},\ cos\ \varphi = \frac{B}{a_0}$$

so the above expression can be rewritten as-

$$x = e^{-bt}[\ a_0\ \{cos\ \omega t\ .\ sin\ \varphi + sin\ \omega\ t\ .cos\ \varphi\}]$$

$$= a_0 e^{-bt}\ sin\ (\omega t + \varphi)$$

$$or\ \ x = a_0 e^{-b\frac{t}{2x}} \sin(\omega t + \varphi)$$

This is the equation of a damped harmonic oscillator with amplitude a_0e^{-bt} $or\ x = a_0 e^{-b\frac{t}{2x}}$.

The sine term in the equation suggests that the motion is oscillatory whereas, the exponential term implies that the amplitude is decreasing gradually.

Therefore, we may conclude that the damping produces two effects:

(i) The frequency of damped harmonic oscillator, $\frac{\omega}{2\pi}$ is smaller than its natural frequency $\frac{\omega_0}{2\pi}$, or damping somewhat decreases the frequency or increases the time period of oscillator.

(ii) The amplitude of the oscillator does not remain constant at a_0, which represents amplitude in the absence of damping, but decays exponentially with time, according to the value of term e^{-bt}.

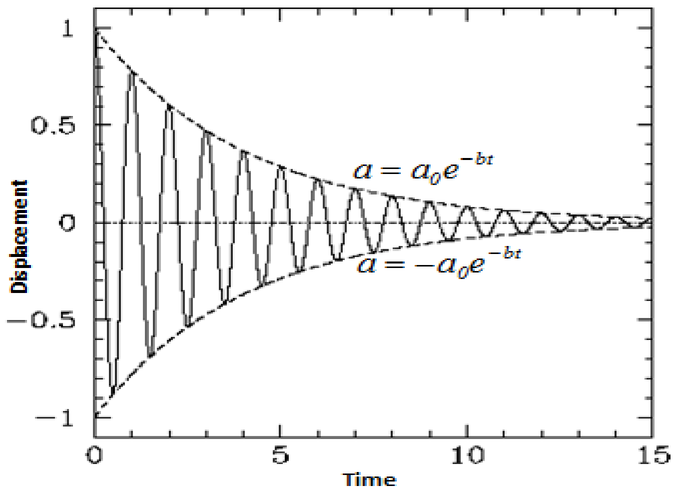


Figure 9: Decay of amplitude of a damped harmonic motion with time

5.9.2 CHARACTERIZING WEAK DAMPING

1 RELAXATION TIME, τ

It refers to the time in which the amplitude of a weakly damped system reduces to 1/e times of the original value. In other words, it is the time in which the mechanical energy of an oscillator decays to 1/e times its initial value.

The energy of a damped harmonic oscillator is given by

$$E = E_0 e^{-\frac{t}{\tau}}$$

Here E_0= the initial value of energy

E= Energy at time t

$$\text{At } t = \tau,\ E = \frac{E_0}{e} \tag{1.37}$$

2 LOGRATHMIC DECREMENT

Due to damping, the amplitude of a damped harmonic oscillator decreases exponentially with time. Suppose that a_n and a_{n+1} be the two successive amplitudes of the oscillations of the particles on two sides of the equilibrium position respectively. The time interval between these two successive amplitudes clearly would be T/2 - half the time period (T) of oscillations. We can further write-

$a_n = a_{0e}$-bkt

and $a_{n+1} = a_0 e^{-b(t+\frac{T}{2})}$

Therefore, $$\frac{a_n}{a_{n+1}} = e^{\frac{bt}{2}} = d \qquad (1.38)$$

Here d is a constant, and it refers to decrease in successive amplitudes. It is known as the decrement for that motion.

Further, on taking the natural log of Eq.(1.38), we obtain

$$\ln d = \frac{kT}{2} = \lambda \qquad (1.39)$$

or, $d = e^{\lambda}$

The constant λ, which is the natural logarithm of decrement or the ratio between two successive amplitudes of the oscillations, is referred to as logarithmic decrement for that oscillatory motion.

3 QUALITY FACTOR

As the name suggests, quality factor is a measures the quality of a harmonic oscillator, as far as damping is concerned.“Lesser the damping, better will be the quality of harmonic oscillator as an oscillator”. Therefore, an harmonic oscillator with low damping will have high value of its quality factor, Q. It is also referred to as the figure of merit of a harmonic oscillator and is defined as the 2π times the ratio between the energy stored and the energy lost per period. Being a ratio, it is a dimensionless quantity.

Thus $$Q = 2\pi \frac{Energy\ stored}{Energy\ lost\ \ per\ \ period}$$

$$= \frac{2\pi E}{PT}$$ (Here P= Average loss of energy per cycle = $\frac{E}{\tau}$)

And since, $\frac{2\pi}{T} = \omega$, we have

$$Q = \frac{E\omega}{P} = \frac{E\omega}{E/\tau} = \omega\tau \qquad (1.40)$$

In case of low damping, $\omega = \omega_0$ and we can rewrite the above equation as

$$Q = \omega_0 \tau$$

But, as we know

$$\omega_0 = \sqrt{\frac{k}{m}} \text{ and } \tau = \frac{m}{\gamma}, \text{ so that,}$$

$$Q = \sqrt{\frac{k}{m}}.\frac{m}{\gamma} = \sqrt{\frac{km}{\gamma}} \qquad (1.40a)$$

Clearly, if γ is small (i.e. if the damping is low), the value of Q will be large.

Further, the energy of a damped harmonic oscillator is

$$E = E_0 e^{-\frac{t}{\tau}} \qquad (1.40b)$$

Hence, at $t = \tau$, we have

$$E = E_0 e^{-1} = \frac{E_0}{e}$$

Example 10: A Harmonic oscillator is represented by the equation

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + kx = C$$

With m = 0.25 kg, γ =0.070 kg/s and k =85 N/m, calculate the period of oscillation.

Solution 10: The period of oscillation of a damped oscillator is given as $T = \frac{2\pi}{\sqrt{\frac{k}{m} - \left(\frac{\gamma}{2m}\right)^2}}$

$$= \frac{2\pi}{\sqrt{\frac{85}{0.25} - \left(\frac{0.07}{2\times 0.25}\right)^2}}$$ =**0.34 seconds**

Example 11: For the harmonic oscillator given in problem 1, calculate (i) the number of oscillations in which its mechanical energy will drop to one-half of its initial value. Also calculate its quality factor.

Solution 11- The average energy associated with a damped harmonic oscillator is given by

$$< E >= E_0 e^{-2bt} = E_0 e^{-\frac{\gamma t}{m}}$$

$$\frac{< E >}{E_0} = e^{-\frac{\gamma t}{m}}$$

$$for \frac{< E >}{E_0} = \frac{1}{2} \text{ , } we\, have$$

$$e^{-\frac{\gamma t}{m}} = \frac{1}{2}$$

Taking natural algorithm on both sides and rearranging the terms, we can rewrite it as

$$t = \frac{m \ln 2}{\gamma} = \frac{0.25 \times 0.693}{0.070} = 2.48s$$

No of oscillation in this time interval= $\frac{2.48}{0.34} \cong 7 oscillations$

(ii) The quality factor Q is given as

$$Q = \frac{\omega_d \tau}{2} \cong \frac{\omega_0 m}{\gamma}$$

Since $\omega_d = \omega_0$ and $\tau = \frac{1}{b} = \frac{2m}{\gamma}$

$$Q = \frac{18.43 \times 0.25}{0.07} = 66$$

Example 13: The amplitude of a damped harmonic oscillator reduces from 25 cm to 2.5 after 100 complete oscillations, each of period 2.3 seconds. Calculate logarithmic decrement of the system.

Solution 13: Here, the amplitude ration of oscillation separated by 100 oscillations is

$$= \frac{25cm}{2.5cm} = 10$$

Therefore, logarithmic decrement $= \frac{1}{100} \ln 10 = \frac{2.3}{100} = 0.023$

5.10 FORCED HARMONIC OSCILLATOR

A damped harmonic oscillator on which an external periodic force is applied is called a forced damped harmonic oscillator. Such an oscillator is also called a driven harmonic oscillator. In such an oscillator, the frequency of the externally applied periodic force is not necessarily the same as the natural frequency of the oscillator. In such a case, there is a sort of tussle between the damping forces tending to retard the motion of the oscillator and the externally applied periodic force which tend to continue the oscillatory motion. As a result, after some initial erratic movements, the oscillator ultimately succumbs to the applied or the driving force and settles down to oscillating with the driving frequency and a constant amplitude and phase so long as the applied force remains operative.

5.10.1 DIFFERENTIAL EQUATION FOR FORCED DAMPED HARMONIC OSCILLATOR

When an external periodic force *F(t)* is applied to a damped harmonic oscillator, the differential equation for the oscillator will have one additional term for the applied time dependent periodic force and we can write

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x + F(t) = 0 \qquad (1.41)$$

If the applied external force is represented as $F(t) = f cos(nt)$, where f and n are constants, then Eq. (1.41) becomes

$$m\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = f\cos(nt) \qquad (1.42)$$

We can further simplify the equation as

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega_0^2 x = \mathrm{a}\cos(nt) \qquad (1.43)$$

where a = f/m, $2b = \gamma$ and $\omega_0^2 = k/m$ is the natural frequency of the oscillator

Eq. (1.43) represents the differential equation for damped forced harmonic oscillator.

5.10.2 SOLUTION OF DIFFERENTIAL EQUATION

You may note that the differential equation (Eq. (1.43) for the damped forced harmonic oscillator is a linear inhomogeneous second order ordinary differential equation; the inhomogeneous term is represented by the externally applied time dependent periodic force.

The solution to this equation comprises two parts: the general solution and the particular solution. Let us learn about them now.

5.10.3 GENERAL SOLUTION

The general solution of the differential equation for damped forced harmonic oscillator comprises of two terms - one representing the homogeneous ordinary differential equation part and the other representing the particular integral part.

If X_i is a particular solution of an inhomogeneous differential equation, and X_n is a solution of a complementary homogeneous equation then $X(t) = X_i(t) + X_n(t)$ is a general solution.

Thus, the general solution of this linear inhomogeneous ODE can be expressed as

$$x(t) = x_H(t) + x_p(t) \qquad (1.44)$$

$x_H(t)$ is the solution of the corresponding homogeneous part of the equation. The homogeneous part is same as the differential equation for the solution of damped harmonic oscillator and its solution is given as

$$x_H(t) = \frac{1}{2}a_0 e^{-\lambda t}\left[\left(1 + \frac{\lambda}{\sqrt{\lambda^2 - \omega_0^2}}\right) e^{\sqrt{(\lambda^2 - \omega_0^2\ )}t} + \left(1 - \frac{\lambda}{\sqrt{\lambda^2 - \omega_0^2}}\right) e^{-\sqrt{(\lambda^2 - \omega_0^2\ )}t}\right] \qquad (1.45)$$

To obtain the particular solution, $x_p(t)$, let us assume the solution of the form

$$x_p(t) = A cos(nt - \emptyset) \qquad (1.46)$$

Ø is the possible phase difference between the applied force and the displacement of the oscillator and n is the frequency of the applied force.

Now we have to obtain dx/dt and d^2x/dt^2 and substitute in Eq. (1.43). We have

$$\frac{dx}{dt} = -An sin(nt - \emptyset)$$

$$\frac{d^2x}{dt^2} = -An^2 cos(nt - \emptyset)$$

Substitution in Eq. (1.43) gives

$$-An^2\cos(nt - \emptyset) - 2\lambda An\sin(nt - \emptyset) + A\omega_0^2\cos(nt - \emptyset) = \mathrm{a}\cos[(nt - \emptyset) + \emptyset] \qquad (1.47)$$

Expanding the R.H.S gives

$$-An^2\cos(nt - \emptyset) - 2bAn\sin(nt - \emptyset) + A\omega_0^2\cos(nt - \emptyset)$$
$$= a[\cos(nt - \emptyset)\cos\emptyset - \sin(nt - \emptyset)\sin \qquad (1.48)$$

Rearranging we get

$$A(\omega_0^2 - n^2)\cos(nt - \emptyset) - 2bAn\sin(nt - \emptyset) = a[\cos(nt - \emptyset)\cos\emptyset - \sin(nt - \emptyset)\sin\emptyset$$

If this equation is to hold true, then the coefficient of cos(nt – Ø) and sin(nt – Ø) on either sides must be equal

i.e. $A(\omega_0^2 - n^2) = a\cos\emptyset \;\; and \;\; 2bn = a\sin\emptyset$

Squaring and adding these two we get

$$A^2(\omega_0^2 - n^2)^2 + 4b^2n^2 = a^2 \qquad (1.49)$$

Hence

$$A^2 = \frac{a^2}{(\omega_0^2 - n^2)^2 + 4b^2n^2} \qquad (1.50)$$

The amplitude of driven or forced oscillator is given as

$$A = \frac{a}{\sqrt{(\omega_0^2 - n^2)^2 + 4b^2n^2}} \qquad (1.51)$$

We have taken only the positive value of the square root. The negative value will mean opposite phase but then Ø will also change by π and there would, therefore be no effect on the value of *A*. Further, the phase is given by

$$\tan\emptyset = \frac{2bn}{(\omega_0^2 - n^2)} \qquad (1.52)$$

The particular solution of Eq. (1.43) is thus given by

$$x_p(t) = \frac{a}{\sqrt{(\omega_0^2 - n^2)^2 + 4b^2n^2}}\cos(nt - \emptyset) \qquad (1.53)$$

Thus, we can write the general solution as

$$x(t) = \frac{1}{2}a_0e^{-bt}\left[\left(1 + \frac{b}{\sqrt{b^2 - \omega_0^2}}\right)e^{\sqrt{(b^2 - \omega_0^2)}t} + \left(1 - \frac{b}{\sqrt{b^2 - \omega_0^2}}\right)e^{-\sqrt{(b^2 - \omega_0^2)}t}\right]$$
$$+ \frac{a}{\sqrt{(\omega_0^2 - n^2)^2 + 4b^2n^2}}\cos(nt - \emptyset) \qquad (1.54)$$

Where $\frac{a_0}{2}$ and Ø need to be determined by initial conditions.

5.10.4 STEADY STATE SOLUTION

When the tussle between the damping and the externally applied forces ends and the oscillator has settled down to oscillate with the frequency of the applied periodic force, it is said to be in the steady state. In the steady state, the homogeneous term vanishes as t → ∞ whereas the particular solution does not. Thus we have a distinction between the transient state , which is a function of the initial conditions, and a steady state , which depends on the external force. Thus, we can write the steady state solution as

$$x(t) = \frac{a}{\sqrt{(\omega_0^2 - n^2)^2 + 4b^2n^2}} \cos(nt - \emptyset) \qquad (1.55)$$

$$x(t) = A\cos(nt - \emptyset) \qquad (1.56)$$

where $A = \frac{a}{\sqrt{(\omega_0^2 - n^2)^2 + 4b^2n^2}} = \frac{f}{m\sqrt{(\omega_0^2 - n^2)^2 + 4b^2n^2}}$ (1.57)

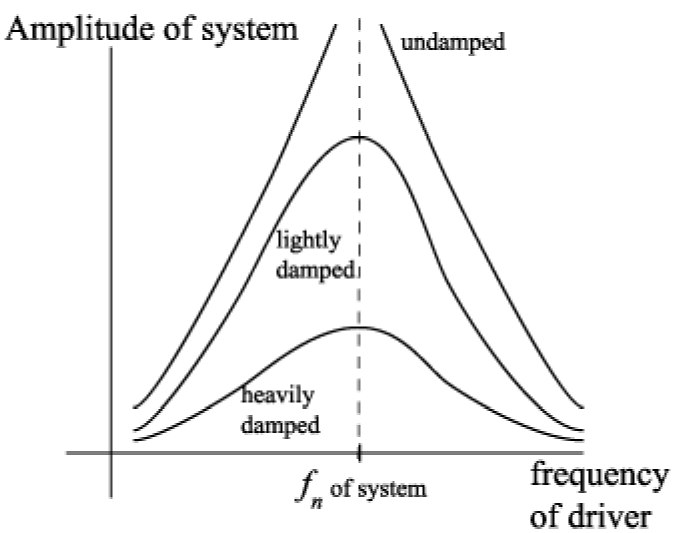


Figure 10: The variation of amplitude of oscillation of a forced oscillator with the frequency of the externally applied periodic force

5.10.5. EXAMPLE OF FORCED OSCILLATIONS - A DRIVEN LCR CIRCUIT

Consider an LCR circuit consisting of an inductor, *L*, a capacitor, *C*, and a resistor *R*, connected in series with a sinusoidal voltage source, *V(t)*, as shown in Fig 2 below.

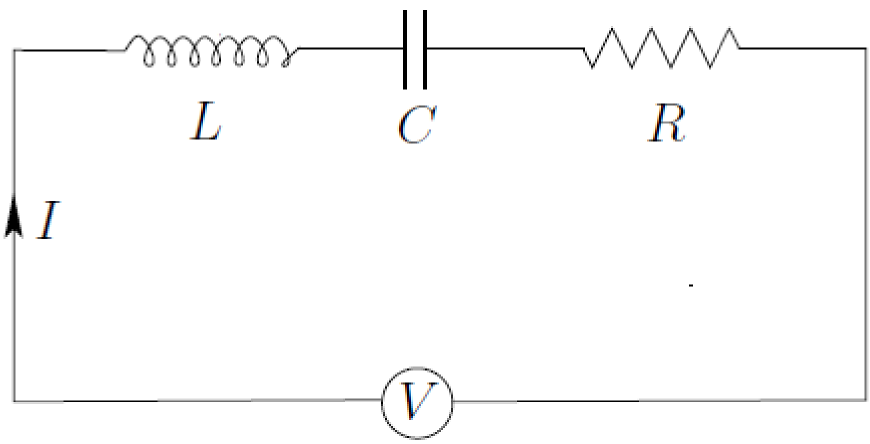


Figure 11: A driven *LCR* circuit

Let *I(t*) be the instantaneous current flowing through this circuit, Now, as per Kirchoff's second circuital law, the sum of the potential drops across the various components of a closed circuit loop is equal to zero. Thus, since the potential drop across an emf is *minus* the associated voltage, we obtain

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = V \qquad (1.58)$$

Where $\frac{dQ}{dt} = I$ and $\frac{d^2Q}{dt^2} = \dot{I}$. Suppose that the emf is such that its voltage oscillates sinusoidally at the angular frequency ω (> 0) with a peak value V_0 (> 0) so that

$$V(t) = V_0 \sin(\omega t) \qquad (1.59)$$

Substituting (1.59) in (1.58), we get

$$L\dot{I} + RI + \frac{Q}{C} = V_0 \sin(\omega t) \qquad (1.60)$$

Dividing equation (1.60 21) by L and differentiating with respect to time, we get

$$\ddot{I} + \gamma\dot{I} + \omega_0^2 I = \frac{\omega V_0}{L}\cos(\omega t) \qquad (1.61)$$

Where $\omega_0 = \frac{1}{\sqrt{LC}}$ and $\gamma = \frac{R}{L}$

Equation (1.61) is similar to the differential equation representing a driven damped harmonic oscillator. The current driven in the circuit by the oscillating emf is given as

$$I(t) = I_0 \cos(\omega t - \emptyset) \qquad (1.62)$$

where

$$I_0 = \frac{\omega V_0}{L\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}} \qquad (1.63)$$

$$\emptyset = tan^{-1}\left(\frac{\gamma\omega}{\omega_0^2 - \omega^2}\right) \qquad (1.64)$$

In the expression for I_0, the denominator $\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$ functions as the effective resistance in the circuit. It is called the impedance of the circuit.

5.10.6 RESONANCE

In general, resonance may be defined as a tendency of a vibrating / oscillating system to respond most strongly to a driving force whose frequency is close to its own natural frequency of vibration / oscillation.

For a weakly damped forced (driven) oscillator, after a transitory period, the object will oscillate with the same frequency as that of the driving force. The plot of amplitude $x(\omega)$ versus angular frequency is shown in Fig. 3 below. If the angular frequency is increased from zero, the amplitude, $x(\omega)$ will increase until it reaches a maximum when the angular frequency of the driving force is the same as the natural frequency of the undamped oscillator. This phenomenon is called resonance.

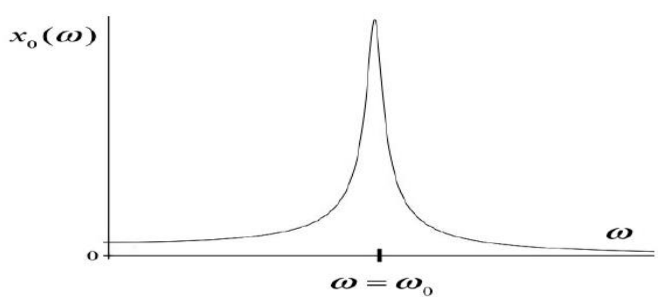


Fig. 12: Plot of amplitude $x(\omega)$ with driving angular frequency ω of a weekly damped harmonic oscillator

From Eq.(18), it is clear that, for a damped forced harmonic oscillator, the amplitude of the oscillator in the steady state depends not only on the amplitude of the driving force, but also on the relation between the frequency, n of the driving force and the natural frequency, ω of the oscillator, as well as on the damping parameter b.

For $n \to 0$ we have $A \to a/\omega$. For $n \to \infty$ we obtain $A \to 0$. In between these two extremes, the amplitude may reach a maximum which we refer to as the resonance frequency.

To obtain an expression for resonance frequency, we differentiate the denominator of Eq. (1.57) with respect to n and then equate it to zero.

$$\frac{d}{dn}[\omega_0^2 - n^2)^2 + 4b^2n^2] = -4n[\omega^2 - n^2] + 8b^2n = 0$$

The non - trivial solution is:

$$n = n_r = \sqrt{\omega_0^2 - 2b^2} \tag{1.65}$$

This is the resonance frequency.

As we have already studied that resonance is defined mathematically using the differential Eq.(2 6) for a forced driven harmonic oscillator where the resonance is defined as the existence of a solution that is unbounded as $t \to \infty$. This corresponds to what we call as pure resonance. It occurs exactly when the natural internal frequency matches the natural external frequency, in which case all solutions of the differential equation are unbounded.

The notion of pure resonance is easy to understand both mathematically and physically, because frequency matching characterizes the event. This ideal situation never happens in the physical world, because damping is always present. In the presence of damping only bounded solutions exist for Eq. (1.65 26).

5.11 SUMMARY

In this unit, we have studied about what is meant by the periodic motion, the oscillatory motion and SHM. We studied about the restoring force that comes in to play due to the displacement from the mean or the equilibrium position and how the restoring force is proportional to the magnitude of the displacement in case of SHM. We studied the two simple systems, simple pendulum and spring-mass system, which are both examples of SHM. Using the knowledge of Newton's second law of motion, we wrote the equation of motion for SHM and derived the solution of the differential equation used to describe SHM.

This unit also describes damped and forced harmonic oscillator. We studied that the differential equation for damped and forced harmonic oscillator is a second order non homogeneous linear ordinary differential equation. We obtained the differential equation and discussed the solution which has two components: one is the general solution and the other being the steady state solution. The general solution has two parts. Further, the steady state solution is obtained in the time domain $t \to \infty$. For the forced oscillator in steady state, we studied the concept of resonance.

5.12 GLOSSARY

Displacement – net change in location of a moving body; in case of SHM, it is measured from the equilibrium position.

Force – anything that can change the state of motion of an object.

Frequency – the number of complete cycles per second made by a vibrating object.

Hooke's Law – the extension of a spring is proportional to the tension in the spring.

Velocity – speed in a given direction.

Wavelength – the distance between two adjacent wave-crests.

Angular acceleration – it is the rate of change of angular velocity. In SI units, it is measured in (rad/s^2), and is usually denoted by the Greek letter alpha (α).

Angular amplitude – it is the maximum angle (disregarding the direction) that a rotating body goes through from the equilibrium position

Angular displacement – it is the angle that a rotating body goes through.

Angular velocity – it is defined as rate of change of angular displacement. SI units is (rad/s).

Kinetic energy – energy of an object due to motion.

Potential energy – energy due to position.

Mechanical energy – it is the sum of the kinetic energy and the potential energy.

Dissipative- continuously loosing energy / amplitude

Damping: reduction in the amplitude of oscillation as a result of energy being drained from the system to overcome frictional or other resistive foeces

Overdamped – Having high value of damping effects

Underdamped- Having small value of damping

Friction- resistance

Driven or forced Oscillator- An oscillator to which an external periodic force is applied

Transient state- The state of the driven harmonic oscillator prior to achieving the steady state.

Steady state- The state of the harmonic oscillator which is independent of the initial state and depends only upon the driving frequency and the damping ratio.

Resonance- The condition when the oscillator under the influence of external driving force oscillates with greater amplitude at a specific preferential frequency.

5.13 REFERENCES

1. Concepts of Physics, Part I, H C Verma – Bharati Bhawan, Patna
2. The Physics of Waves and Oscillations, N K Bajaj – Tata McGraw-Hill, New Delhi
3. Fundamentals of Physics, David Halliday, Robert Resnick, Jearl Walker – John Wiley & Sons
4. Physics, Jim Breithaupt – Palgrave

5.14 SUGGESTED READINGS

1. Fundamentals of Physics, David Halliday, Robert Resnick, Jearl Walker – John Wiley & Sons
2. Berkeley Physics Course Vol 3, Waves, C Kittel et al, McGraw- Hill Company

5.15 TERMINAL QUESTIONS

1. A horizontal spring-mass system of spring constant k and mass M executes SHM with frequency ν. When the block is passing through its equilibrium position, an object of mass m is put on it and the two move together. Find the new frequency of vibration.

2. A particle executes SHM with amplitude of 0.5 cm and frequency of 100 s^{-1}. What is the maximum speed of the particle?

3. A weight suspended from a spring oscillates up and down. The restoring force in the weight is zero at (a) highest point, (b) lowest point, (c) middle point, (d) none of these.

4. A person goes to bed at sharp 10:00 pm every day. Is it an example of periodic motion? If yes, what is the time period? If no, why?

5. A particle moves on the x-axis according to the equation $x = A + B \sin \omega t$. Is the motion SHM? If yes, what is the amplitude?

6. Select the correct statement(s). More than one choice may be correct.

(a) A simple harmonic motion is necessarily periodic.
(b) A simple harmonic motion is necessarily oscillatory.
(c) An oscillatory motion is necessarily periodic.
(d) A periodic motion is necessarily oscillatory.

7. Write notes on:

(i) SHM (ii) Spring-Mass System (iii) Time period (iv) Angular Frequency

8. For a particle executing SHM along x-axis, the restoring force is given by
(a) $-Akx$ (b) $A\cos kx$ (c) $A\exp(-kx)$ (d) Akx

9. The potential energy of a particle executing SHM is given by
(a) $U = k/2(x-a)^2$ (b) $U = kx + kx^2 + kx^3$ (c) $U = A\exp(-bx)$ (d) $U = constant$

10. A particle executes simple harmonic motion of amplitude A along the x-axis. At $t = 0$, the position of the particle is $x = A/2$ and it moves along the positive x-direction. Find the phase constant if the equation is written as $x(t) = A\sin(\omega t + \delta)$.

11. A body of mass 2 kg, suspended through a vertical spring, executes SHM of period 4 s. If the oscillations are stopped and the body hangs in equilibrium, find the potential energy stored in the spring. [$g = 10\ ms^{-2}$]

12. The work done by the spring-mass system during one complete oscillation is equal to

(a) The total energy of the system
(b) Kinetic energy of the system
(c) Potential energy of the system
(d) Zero

13. A particle of mass m is hanging vertically by an ideal spring of force constant k. If the mass is made to oscillate vertically, its total energy is

(a) maximum at the extreme position
(b) maximum at the mean position
(c) minimum at the mean position
(d) none of the above

14. Write short notes on:

(i) Acceleration in SHM (ii) Energy Variation in SHM
(iii) Phasor model of SHM

15. Why does the amplitude of oscillations go on decreasing in case of damped harmonic oscillator? Assuming damping to be proportional to the velocity, find an expression for the frequency of oscillations.
16. A system executing damped harmonic motion is subjected to an external periodic force. Investigate the forced vibration and obtain the condition of resonance.

17. Show that the ratio of two successive maxima in the displacement of a damped harmonic oscillator is constant.
18. If the amplitude of a damped harmonic oscillator decreases to 1/e of its initial value after n($\gg$ 1) periods, show that the ratio of the period of oscillation to the period of the oscillation with no damping is

$$\left(1+\frac{1}{4\pi^2 n^2}\right)^{\frac{1}{2}} \approx 1+\frac{1}{8\pi^2 n^2}$$

19. A spring-mass system is subjected to restoring and frictional forces of magnitude kx and $\gamma \frac{dx}{dt}$ respectively. It oscillates with a frequency of 0.5 Hz. Its amplitude reduces to half in 2 seconds. Calculate the damping coefficient γ and spring constant k, in terms of mass, m. Also write the differential equation of motion.

20. The quality factor of a tuning fork of frequency 512 Hz is 6 x 10^4. Calculate the time in which its energy drops to $E_0 e^{-1}$. How many oscillations will the tuning fork make in this time.

21. A particle of mass m moves under the influence of external periodic force F sin pt along x axis in addition to the restoring force $-kx$ (also along x - axis) and damping force $-\beta x i$ along x axis . Set up the differential equation of motion and find the steady-state solution.

22. Show that in case of a system undergoing a forced oscillation, the response is independent of its mass if $n << \omega_0$ and is independent of spring constant if $n >> \omega_0$

23. A damped harmonic oscillator consists of a block (m = 2 kg), a spring (k = 30 N/m), and a damping force (F = -bv). Initially, it oscillates with amplitude of 25 cm; because of the damping, the amplitude falls to three-fourths of this initial value at the completion of four oscillations.
(a) What is the value of b?
(b) How much energy has been "lost" during these four oscillations?

Answers of Selected Terminal Questions:

1. Original frequency of SHM,

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k}{M}}$$

The new frequency of SHM,

$$\nu_{new} = \frac{1}{2\pi}\sqrt{\frac{k}{m+M}}$$

Therefore,

$$\nu_{new} = \nu\sqrt{\frac{M}{m+M}}$$

2. $|v|_{max} = \omega A = (2\pi \times 100)(0.5 \times 10^{-2}) = \pi\ m/s$

3. (c) because at the equilibrium or mean position the restoring force is zero.

4. Yes. Time period = 24 hours.

5. No. SHM is a special case of oscillatory motion, where a body moves back and forth repeatedly about a fixed position. Here nothing like that happens!

6. (a), (b) 8. (a), 9. (a)

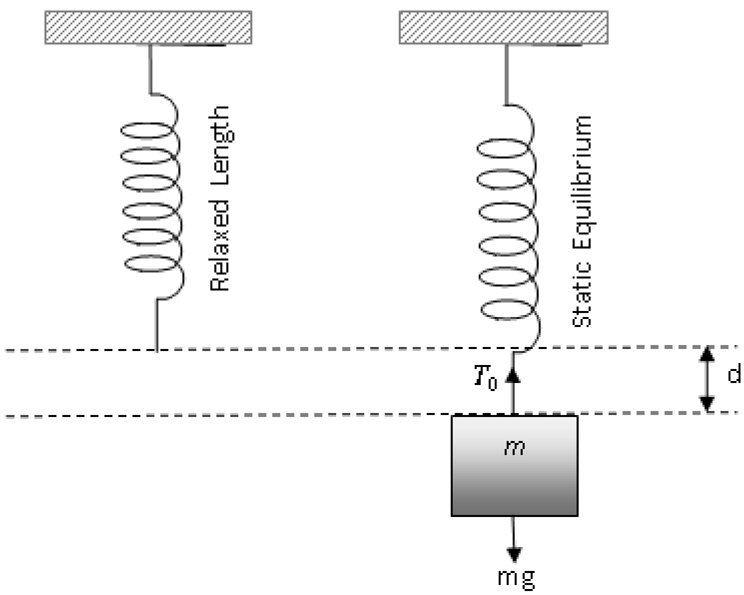
10. At $t = 0, x = A/2$. Therefore, $A/2 = A\sin\delta$ or $\delta = \pi/6$ or $5\pi/6$. The velocity is given by

$$\frac{dx}{dt} = v(t) = A\omega\cos(\omega t + \delta)$$

At $t = 0, v = A\omega\cos\delta$. Now, $\cos\pi/6 = \sqrt{3}/2$ and $\cos 5\pi/6 = -\sqrt{3}/2$. We are given that the velocity is positive at $t = 0$, therefore the phase constant cannot be $5\pi/6$.

$$\therefore \delta = \pi/6$$

11. The body hangs in equilibrium at the end of a spring as shown below.



Now the potential energy stored in the equilibrium position in the spring will be because of the elongation in the spring, i.e.

$$U = \frac{1}{2}m\omega^2 d^2$$

So, now our task is to calculate *d*. We know the frequency of the oscillation and the mass m of the block. By applying force balance, we have

$$T_0 = mg$$

Also, we know that the tension provides the restoring force,

$$T_0 = -kd$$

From the two equations, we get

$$d = \frac{mg}{k} = \frac{mg}{m\omega^2} = \frac{g}{\omega^2}$$

Therefore, the potential energy will be

$$U = \frac{1}{2}m\omega^2\left(\frac{g}{\omega^2}\right)^2 = \frac{1}{2}m\frac{g^2}{\omega^2}$$

$$= \frac{1}{2}m\frac{g^2}{\left(\frac{2\pi}{T}\right)^2} = \frac{1}{2}(2)\frac{(10)^2}{\left(\frac{2\pi}{4}\right)^2}$$

$$\therefore U = 40.5\,J$$

12. (d) , 13. (d) The total energy remains constant.

19. $\gamma = 0.693$, k= 9.98, $\frac{d^2x}{dt^2} + 0.693\frac{dx}{dt} + 9.98x = 0$,
20. t= 18.7 s, n = 9570,
23: (a) We assume that b is small compared to √km and we take T = 2π/√(m/k) ≈ 1.62 s. It is given that at t = 4T, the amplitude falls to 3A/4, i.e.
$e^{-bt/2m} = 3/4$
-2bT/m = ln(3/4)
or b = 0.18 kg/s.
(b) Energy lost during these four oscillations = ½ k(A^2 – $(3A/4)^2$) = $7kA^2/32$ = 0.410 J

UNIT 6 WAVE MOTION

Structure

6.1 INTRODUCTION

The transportation of energy through the disturbance in medium or through electromagnetic vectors is called waves. The waves transfer energy but there may not be any transportation of matter in the process. For example, when a violinist plays violin, its sound is heard at distant locations. The *sound waves* carry with them energy, with which they are able to move the diaphragm of the ear. When a stone is dropped in the still water in a lake, ripples are formed on the surface of the water body and the *water waves* move steadily in the outward direction. *Electromagnetic waves* are vibrating electric and magnetic fields that travel through space without the need for a medium. The electromagnetic waves include the visible light that, for example, comes from a bulb in our houses and the radio waves that come from a radio station. The other types of electromagnetic waves are microwaves, infrared light, ultraviolet light, X-rays and gamma rays. *Seismic waves* are vibrations of the earth, which become quite significant in the events such as earthquakes.

Although, these various processes of transport of energy are different yet they have a common feature, which we shall from now on refer to as the wave motion. In simple terms, we can say that the wave motion involves the transfer of disturbance (energy) from one point to the other with particles of the medium oscillating about their mean positions. The particles themselves oscillate only over a short distance about their initial positions, and as a result a wave moves through the medium. The medium as a whole does not go in the direction of the motion of the wave.

In the present unit, we will learn about wave motion including the formation and propagation of waves, characteristic features of a wave and the distinction between longitudinal and transverse waves.

6.2 OBJECTIVES

After studying this unit, you should be able to understand

- the meaning of wave, its formation and propagation of waves,
- different types of waves, transverse longitudinal waves and their uses,
- representation of a wave graphically at a fixed position and at a fixed time,
- amplitude, wavelength, frequency and speed of a wave,
- wave in a stretched string
- longitudinal waves in different mediums and give examples, and

6.3 WAVE FORMATION AND PROPAGATION

Let us first consider the example of water wave, which is the most familiar kind of wave that we can generate and observe easily. When we drop a stone in a lake or a water tub, we observe circular ripples that spread out from the point where the stone strikes the water surface, as shown in Fig. 1.

Figure 1: Waves generated on the water surface.

Looking at these ripples, you may wrongly get an impression that water moves with them. But, if you observe carefully, you will notice that water actually does not move along with the ripples that are generated. You can easily verify this fact by placing a paper boat or a dry leaf on the water surface and observing how it moves. You will notice that the paper boat or the dry leaf just bounces up and down at the same place on the surface of water and does not move with the ripples. This means that water particles do not have any translational motion. However, water particles do undergo oscillatory motion caused due to dropping of the stone in the still water. The disturbance caused at the point of contact of the stone with water surface is progressively transferred to adjacent water particles due to the oscillatory motion. The term “disturbance” refers to the deformation in the shape of the water surface (or any other medium such as air, string etc.) with respect to its undisturbed surface.

6.3.1 Mechanical Wave:

Mechanical wave can be produced using a thin and long elastic string with its one end fixed to a wall. By holding the other end of the string with your hand so that the string is stretched and taut and quickly moving your hand up and down once, you may observe a disturbance travelling along the length of the string (Figure 2). If you keep your hand moving up and down, you will observe a series of disturbances moving along the string giving rise to a wave.

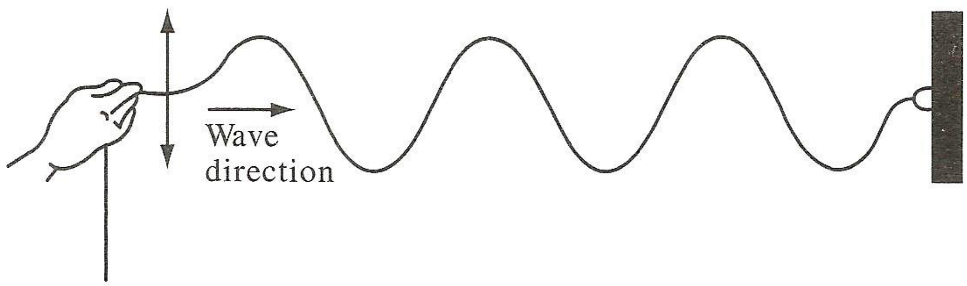


Figure 2: A mechanical wave.

From the above descriptions of waves, one may conclude that:

1- A wave is generated due to two simultaneous, at the same time, distinct motions. The first one is the oscillatory motion of the particles of the medium and the second is the linear motion of the disturbance.
2- In wave motion, the propagation of a disturbance does not take place due to the physical movement of the particles in the medium. The disturbance actually propagates because of the transfer of energy from one particle to the other progressively. Thus, we may conclude that the waves transport energy and not the matter.

The oscillations of the particle of a medium and the propagation of wave in the medium are intimately connected. To appreciate the nature of this relationship, refer to Figure 3, which shows a thin elastic string tied to a spring-mass system executing vertical oscillations. The other end of the string is tied to a rigid support. We assume that the motion of the spring-mass system is without any friction and that the vertical oscillations by the mass are without any lateral movement. Figure 4, further breaks down the waveform shown in Figure 3b and shows the snapshots of the waveform on the string taken at intervals of T/8, i.e. at time t = 0, T/8, T/4, 3T/8, T/2, 5T/8, 3T/4, 7T/8 and T. The arrows attached to each of the nine particles indicate the directions along which these particles are about to move at a given instant. At t = 0, all the particles are at their mean position as shown in Figure 4a.

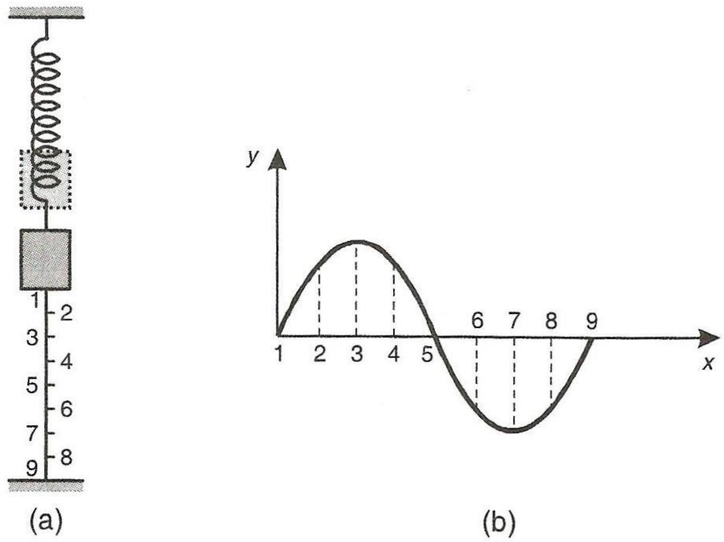


Figure 3: (a) A vertically oscillating spring-mass system fastened to a string, and (b) waveform of the motion of the string.

The particles in the string begins to oscillate due to the transfer of mechanical energy and momentum from the spring-mass system and their motion is sustained due to the elasticity of the medium, in this case string. One particle transfers its energy and momentum to another particle and then it transfers its energy and momentum to the third particle and so on. This process continues as long as the spring-mass system keeps oscillating. When the energy that initially activated particle 1 reaches particle 9 at time T, we say that a wave has been generated in the string. We notice that all the particles in the string oscillate up and down about their respective mean positions with time period T and the wave moves along the string with the same time period.

In our discussion until now, we have considered the propagation of mechanical waves on strings and springs for introducing the wave motion. Mechanical waves require material medium such as water, air, etc. to transfer mechanical energy and momentum from one point to another. Therefore, seismic waves, water waves, sound are all examples of mechanical waves. One should note here that sound waves travelling in air columns and on a string, both are examples of mechanical waves, but there is an important difference between the two. While the former is an example of longitudinal waves, the latter are transverse waves. We will briefly study about these waves in the next section.

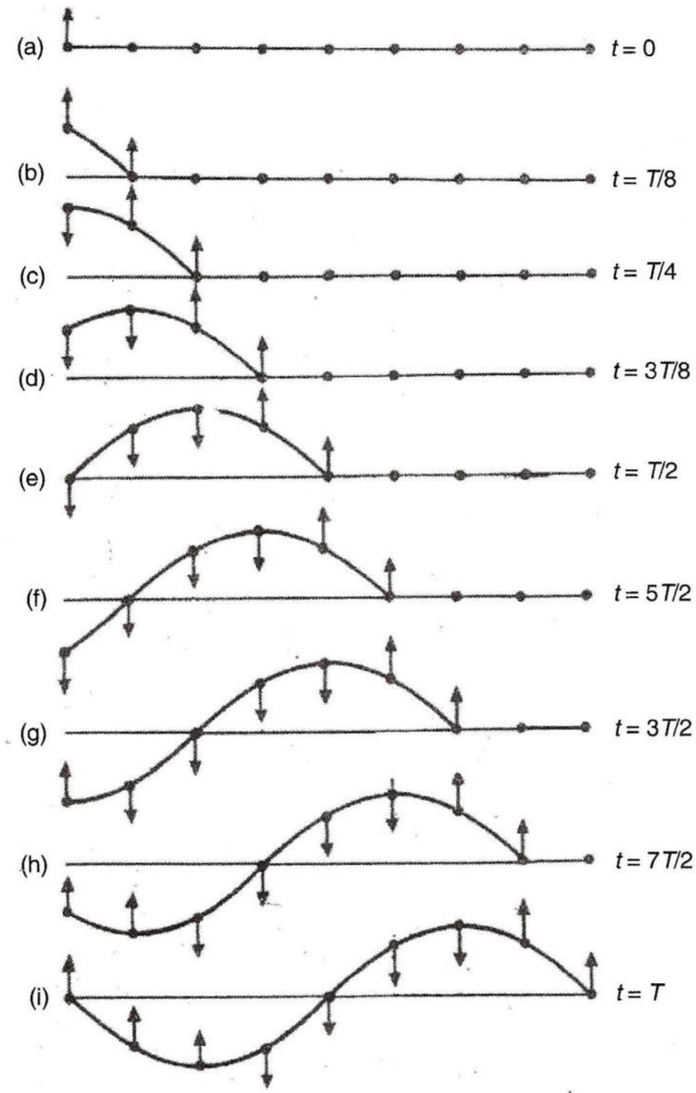


Figure 4: Snapshots of the motion of the particles 1 to 9 in the string beginning at the instant t = 0 and up to the instant t = T at intervals of T/8.

6.3.2 Transverse Waves

In transverse waves, the particles of the medium oscillate perpendicular to the direction in which the wave travels. Travelling waves on a taut string, which we discussed in the previous section, are transverse waves. When the one end of the string is rigidly fixed and the other end is given periodic up and down jerks, the disturbance propagates along the length of the rope but the particles oscillate up and down. The disturbance travels along the rope in the form of crests (upward peak) and troughs (valley) as shown in Figure 3.

Secondary seismic waves are an example of transverse waves. They travel more slowly than the primary seismic waves. Secondary seismic waves shake the material they travel through from side to side. Transverse waves require that there should be a shearing force in the medium. Hence, they can be propagated only in the medium which will support a shearing stress, i.e. mainly solids. For this reason, mechanical transverse waves cannot pass though a liquid because liquid molecules slide past each other. Electromagnetic waves, which do not require any medium to propagate, are also an example of transverse waves. The electric and the magnetic field of an electromagnetic wave vibrate at right angles to the direction of propagation and also at right angles to each other.

8.3.3 Longitudinal Waves

In longitudinal waves, the oscillation of the particles is parallel to the direction in which the wave travels. Disturbance travelling in a spring parallel to its length, a pressure variation propagating in a liquid are examples of longitudinal waves. Longitudinal waves do not require shearing stress and hence can travel in any elastic medium – solid, liquid and gas.

Consider a stretched spring. If one end of the spring is suddenly given an in and out oscillation parallel to the length of the spring, the coils of the spring start exerting forces on each other and the compression and the expansion points travel along the length of the spring. The coils oscillate right and left parallel to the spring as shown in Figure 5. Compressions, which is the crowding together of the molecules, and rarefactions, which is the spreading out of the molecules away from each other, travel along the spring. The pressure at the compression point is higher and the pressure at a rarefaction point is lower.

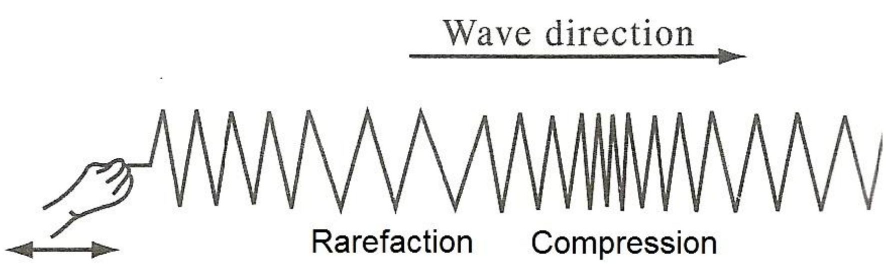


Figure 5: Longitudinal wave generated in a stretched spring.

The spring in the above example can be replaced by a long tube of air with a piston at the left end. The piston is set into oscillation along the length of the tube. The molecules of air oscillate right and left, i.e. parallel to the wave propagation as shown in Figure 6a.

Sound waves are also longitudinal waves as shown in Figure 6b. A loudspeaker supplied with alternating current creates sound waves because the diaphragm of the loudspeaker is forced to move to and fro. The diaphragm compresses the surrounding air in front of it as it moves forward and then it moves back before creating another compression. Effectively, the air which is the medium of propagation in this case, moves to and fro as the sound waves pass through it. Primary seismic waves are another example of longitudinal waves. They travel faster than the secondary waves, and can travel through solids and liquids as they push and pull on the medium they travel though.

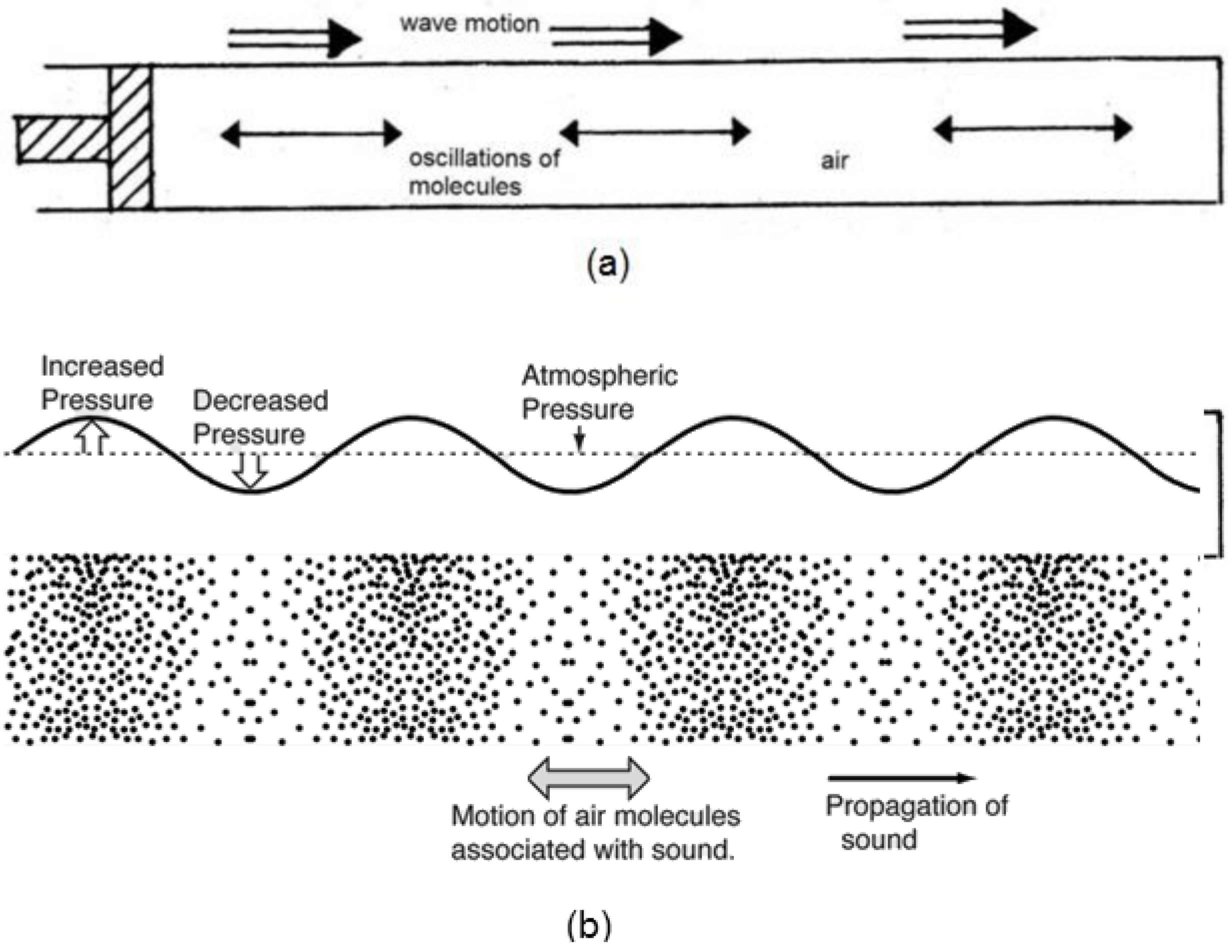


Figure 6: (a) Longitudinal waves generated in a tube of air with a piston at one end. (b) Sound waves in air.

Water waves are a combination of longitudinal and transverse waves. Each particle near the surface moves in a circular orbit, so that a succession of crests and troughs occur. At a crest, the water at the surface moves in the direction of the wave and at trough, it moves in the opposite direction.

Self Assessment Question (SAQ) 1: What type of mechanical waves do you expect to exist in (a) vacuum, (b) air, (c) water, (d) rock?

Self Assessment Question (SAQ) 2: Choose the correct option:

Elastic waves in solid are

(a) Transverse (b) Longitudinal

(c) Either transverse or longitudinal (d) Neither transverse and longitudinal.

Self Assessment Question (SAQ) 3: Give evidence in support of the fact that sound is a mechanical wave.

Self Assessment Question (SAQ) 4: Choose the correct option:

Mechanical waves on the surface of a liquid are

(a) Transverse (b) Longitudinal (c) Torsional (d) Both transverse and longitudinal.

6.4 WAVE PROPERTIES

In the preceding sections, we saw that when a wave moves, the displacements of the particles change with time as well as with the position. In one complete cycle of oscillation, the particles in the medium are displaced in one direction from their mean position to a position of maximum displacement, come back to the mean position and move in the opposite direction to the other extreme, and again come back to their mean position. In the following sections, we will be discussing some of the terms that are useful in characterizing the waves.

6.4.1 Wave Speed

The speed of a wave is the distance it covers in one second. It should be carefully noted that the wave speed is completely different from the particle speed. Particle speed is the speed of the vibrating particles in the medium. On the other hand, wave speed is the speed with which the disturbance (or wave) propagates in the medium.

6.4.2 Wave Frequency

The frequency with which the particles of the medium (through which the wave is passing) oscillate is known as wave frequency. In transverse waves, frequency is the number of crests (or troughs) that pass through a point in one second. In longitudinal waves, frequency is the number of compressions (or rarefactions) that pass through a point in one second. It is denoted by the symbol f. The SI unit of frequency is hertz (Hz), which is equal to 1 cycle per second.

We already know that the wave motion requires a source which moves or vibrates with a particular frequency. So an important point to keep in mind is that the frequency of a wave is a property of the source, not of the medium through which it propagates.

6.4.3 Time Period

The time period of the oscillation of the particles in the medium is the time period of the wave and is depicted in Figure 7. It is denoted by the symbol T. The frequency of a wave is the reciprocal of the time period, i.e.

$$f = \frac{1}{T} \tag{1}$$

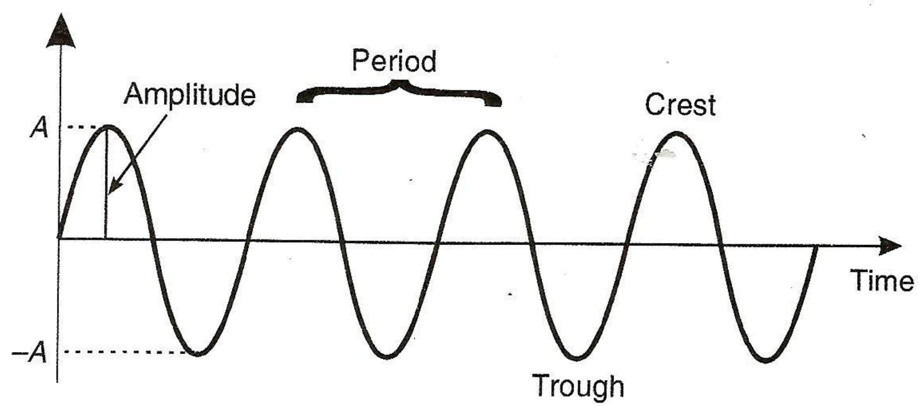


Figure 7: The vibration graph of a wave.

6.4.4 Amplitude

The amplitude of the wave is equal to the maximum positive displacement of the particles from their mean position. Thus, the amplitude of the wave is the same as the amplitude of the oscillating particles. It is depicted in Figures 7 and 8 and is denoted by the symbol A.

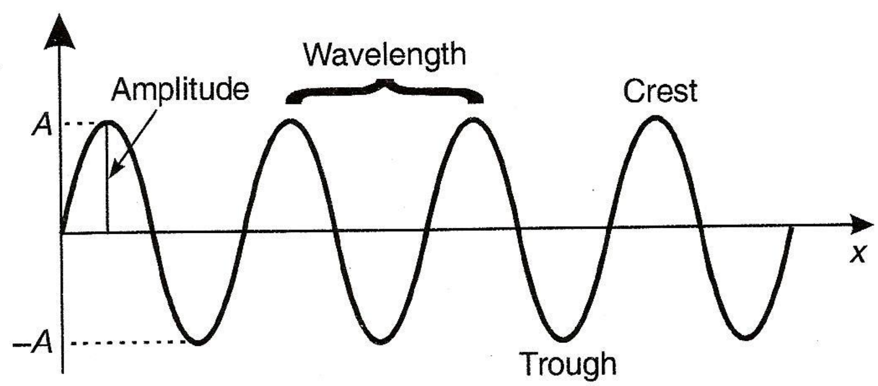


Figure 8: The waveform graph of a wave.

6.4.5 Wavelength

The distance between any two points in the same state of motion defines the wavelength of a wave. Physically, this means that the wavelength is equal to the distance between two consecutive crests (or troughs) and is depicted in Figure 8. Wavelength is denoted by the symbol λ. The wave speed is given by

$$v = \frac{\lambda}{T} \tag{2}$$

Since, the frequency f of a wave is the reciprocal of its period T, the above equation can also be written as

$$v = f\lambda \tag{3}$$

The above equation predicts that in a given medium, the wave speed of a wave of given frequency is constant. Note that equation (3) holds for a transverse as well as a longitudinal wave.

Thus, we can see that the wavelength and the time period represent the spatial and the temporal properties of a wave, respectively. When a wave propagates in a medium, it travels with the same amplitude, time period (or frequency) as those of the particles oscillating in the medium. Hence, we can infer that in a wave, the variation with the position and the time follows the same pattern as that of the oscillating particles. This means that we can represent wave motion both graphically as wells as mathematically. In the graphical representation, the information can be displayed in the following two ways:

1- Keeping the position x fixed and varying the time t.
2- Keeping the time t fixed and varying the position x.

The first type of graph is referred to as the vibration graph of a wave. The vibration graph shows the wave behavior at one position in the path of a wave with time. One can obtain it by fixing a slit at one spot and observing the motion of the wave at different times. Figure 7 shows the vibration graph of a wave. The vibration graph of a wave can be represented as

$$y(t) = A\sin\left(\frac{2\pi t}{T}\right) \tag{4}$$

On the other hand, when the time is kept fixed and the position can vary, the graph obtained is called a waveform graph. It is analogous to a snapshot at any instant of time, such as t = T. A waveform graph displays the wave behavior simultaneously at different locations as shown in Figure 8. We can represent the waveform graph of a wave as

$$y(x) = A\sin\left(\frac{2\pi x}{\lambda}\right) \tag{5}$$

Although, there are similarities in the shapes of vibration and waveform graphs, they should not be confused. While the vibration graph tells us about the shape of the wave, its amplitude and time period, the waveform graph gives us information about the shape of the wave, its amplitude and wavelength.

Example 1: An observer standing at sea coast observes 54 waves reaching the coast per minute. If the wavelength of the waves is 10 m, find the velocity. What type of waves did he observe?

Solution:

Since, 54 waves reach the shore per minute,

$$f = \frac{54}{60} = 0.9\ Hz$$

And as the wavelength of waves is 10 m, therefore,

$$v = f\lambda = 0.9 \times 10 = 9\ m/s$$

The waves on the surface of water are combined transverse and longitudinal waves called ripples. In case of surface waves, the particles of the medium move in elliptical paths in a vertical so that the vibrations are simultaneously back and forth and up and down as shown in Figure 9.

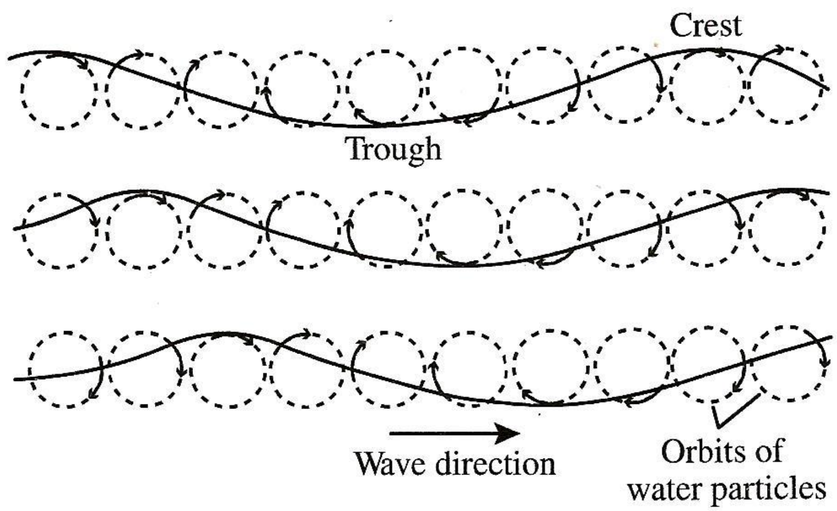


Figure 9: Ripples at different times. At a crest, the surface water moves in the direction of the wave and at trough, it moves in the opposite direction.

Example 2: A light pointer fixed to one prong of a tuning fork touches a vertical plate. The fork is set vibrating and the plate is allowed to fall freely. 8 complete oscillations are counted when the plate falls through 0.1 m. What is the frequency of the tuning fork? Take g = 9.8 m/s^2.

Solution:

Time taken by the plate to fall 0.1 m freely under gravity is given by

$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2(0.1)}{9.8}} = \frac{1}{7}s$$

And in this time, 8 oscillations are recorded on the plate. Therefore, the number of oscillations per second, or in other words, the frequency of the tuning fork will be

$$f = 7 \times 8 = 56\ Hz$$

Example 3: Certain radar emits 9400-MHz radio waves in groups 0.08 μs in duration. The time needed for these groups to reach a target, be reflected and return back to the radar is indicative of the distance of the target. The velocity of these waves, like other electromagnetic waves is $c = 3 \times 10^8\ m/s$. Find

(a) the wavelength of these waves,
(b) the length of each wave group, which governs how precisely the radar can measure distances of the target, and
(c) the number of waves in each group.

Solution:

(a) Since, 1 MHz = 10^6 Hz,

$$9400\ MHz = 9.4 \times 10^9\ Hz$$

Therefore, the wavelength

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8\ m/s}{9.4 \times 10^9\ Hz} = 3.19 \times 10^{-2}\ m$$

(b) The length s of each wave group is

$$s = ct = (3 \times 10^8\ m/s)(8 \times 10^{-8}\ s) = 24\ m$$

(c) There are two ways to find the number of waves n in each group:

$$n = ft = (9.4 \times 10^9\ Hz)(8 \times 10^{-8}\ s) = 752\ waves$$

Or

$$n = \frac{s}{\lambda} = \frac{24\ m}{3.19 \times 10^{-2}\ m} = 752\ waves$$

Self Assessment Question (SAQ) 5: Choose the correct option:

Which of the following cannot travel through vacuum?

(a) Light waves, (b) heat waves, (c) X-rays, or (d) sound waves.

Self Assessment Question (SAQ) 6: A body vibrating with a certain frequency sends waves of wavelength 15 cm in medium A and 20 cm in medium B. If the velocity of wave in A is 120 m/s, that in B will be ____________ m/s.

Self Assessment Question (SAQ) 7: Challenge Question:

An anchored boat is observed to rise and fall through a total range of 2 m once every 4 s as waves whose crests are 30 m apart pass it. Find

(a) the frequency of the waves,
(b) their velocity,
(c) their amplitude, and
(d) the velocity of an individual water particle at the surface.

Self Assessment Question (SAQ) 8: An object oscillates in a simple harmonic motion with a frequency of 100 Hz. Calculate its time period.

Self Assessment Question (SAQ) 9: Sound travels in air with a speed of 332 m/s. The upper limit of audible range is 20,000 Hz. Calculate the corresponding wavelength in cm.

6.5 MATHEMATICAL DESCRIPTION OF WAVE MOTION

If a mathematical equation describes a wave, it must be able to give the position of any particle of the medium at any given instant of time. Consider a transverse wave travelling toward right in a tight string lying on the x-axis. Figure 10 shows the snapshots of a wave travelling along the positive x-axis at the instant t = 0 and at time t. If the wave velocity is v, then as the wave travels the y-coordinate of point C (at x’) at t = 0 is the same as the y-coordinate of point D (at x = x’ + vt) at time t, i.e.

$$y(x,t) = y(x',0) \tag{6}$$

From equation (5), we have

$$y(x',0) = A\sin\left(\frac{2\pi x'}{\lambda}\right) = A\sin\left(\frac{2\pi(x - vt)}{\lambda}\right) \tag{7}$$

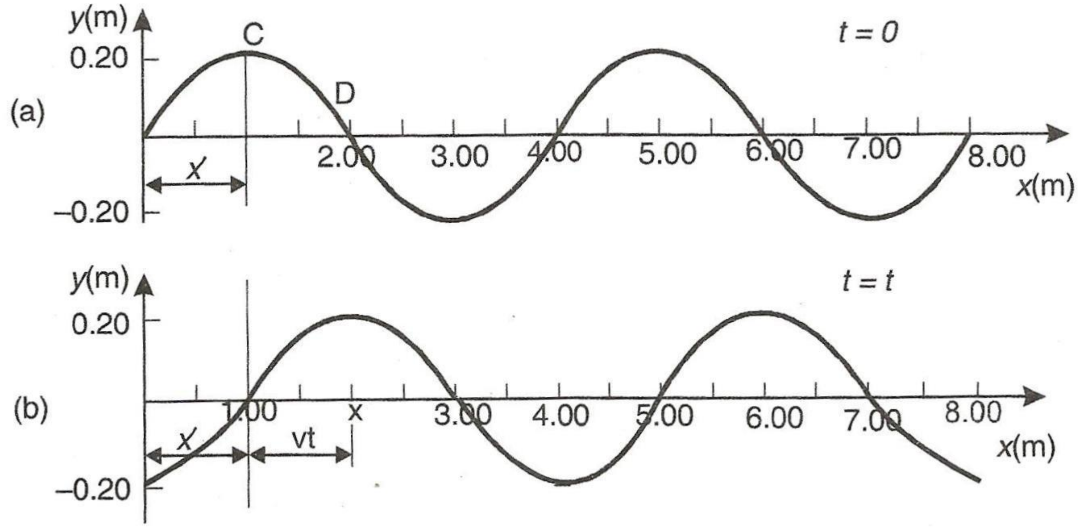


Figure 10

Therefore, from equations (6) and (7), we have

$$y(x,t) = A\sin\left(\frac{2\pi(x - vt)}{\lambda}\right)$$

Replacing v by λ/T in the above equation, the displacement y(x, t) of any particle located at some x-coordinate at any instant of time t is given by

$$y(x,t) = A\sin\left(\frac{2\pi x}{\lambda} - \frac{2\pi t}{T}\right) \tag{8}$$

Equation (8) is known as the wave equation. It can also be written in the following equivalent form:

$$y(x,t) = A\sin(kx - \omega t) \tag{9}$$

where $k = 2\pi/\lambda$ is known as the wave number, which signifies how quickly the wave oscillates is space, and $\omega = 2\pi/T$ is known as the angular frequency, which tells us how quickly the wave oscillates in time.

Also, since the wave velocity is given as λ/T, from equations (8) and (9), we can write

$$v = \frac{\omega}{k} \tag{10}$$

Equation (9) or its other equivalent forms describes a monochromatic wave, since it has a single constant frequency. Note that these equations describe 1-dimensional transverse as well as longitudinal sinusoidal waves travelling in the positive x-direction. This leads to one important difference between the displacement of the particles of the medium and the displacement y(x, t) of any point on the waveform: while the former changes periodically, the latter remains constant. As the wave travels, the entire waveform shifts. Hence, the displacement of a point on the waveform remains the same and this holds for all points on the waveform.

The following equation can easily be derived by replacing v with $-v$, if we want to describe a wave travelling in the negative x-direction.

$$y(x,t) = A\sin(kx + \omega t) \tag{11}$$

Example 4: A wave is represented by

$$y(x,t) = [8\ cm]\sin[(10\ rad/cm)x - (10\ rad/s)t]$$

Determine the amplitude, wavelength, angular frequency, wave number and the velocity of the wave.

Solution:

Comparing the given wave equation with equation (10.9), we find that the wave is travelling in the positive x-direction, with amplitude A = 8 cm, angular frequency $\omega = 10\ rad/s$ and the wave number k = 10 rad/cm.

From the definition of the wave number, we have

$$k = \frac{2\pi}{\lambda}$$

$$\Rightarrow \lambda = \frac{2\pi}{k} = \frac{2\pi}{10} = 0.63\ cm$$

Further, using equation (10.10), we have

$$v = \frac{\omega}{k} = \frac{10\ rad/s}{10\ rad/cm} = 1\ cm/s$$

Example 5: A transverse wave is travelling along a string from left to right. The figure below represents the shape of the string at a given instant. At this instant

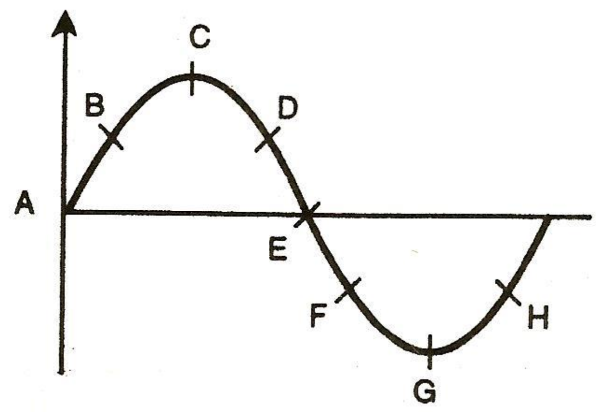


Figure 11

(a) Which points have an upward velocity?
(b) Which points have a downward velocity?
(c) Which points have zero velocity?
(d) Which points have maximum magnitude of velocity?

Solution:

For a wave travelling in positive x-direction, the particle velocity v_p at any instant is given by

$$v_p = \left(\frac{dy}{dt}\right)_x$$

$$\Rightarrow v_p = -\omega A\cos(kx - \omega t) \tag{12}$$

Further, the slope of the wave is given as

$$\frac{dy}{dx} = Ak\cos(kx - \omega t) \tag{13}$$

From equations (12) and (13), we get that the particle velocity v_p is equal to the negative of the product of the wave velocity with the slope of wave at that point,

$$v_p = -\frac{\omega}{k} \times (slope) = -v \times (slope) \tag{14}$$

(a) For upward velocity, $v_p = positive$, so the slope must be negative which is at points D, E and F.
(b) For downward velocity, $v_p = negative$, so the slope must be positive which is at points A, B and H.
(c) For zero velocity, the slope must be zero which is at C and G.
(d) For maximum magnitude of velocity, $|slope| = maximum$ which is at A and E.

Self Assessment Question (SAQ) 10: A simple harmonic wave having an amplitude A and time period T is represented by the equation $y = 5\sin\pi(t+4)\ m$. What are the values of A (in m) and T (in s)?

Self Assessment Question (SAQ) 11: Choose the correct option:

Waves whose crests are 30 m apart reach an anchored boat once every 3 s. The wave velocity (in m/s) is

(a) 0.1 (b) 5 (c) 10 (d) 900

Answer of Selected Self-Assessment Questions (SAQs):

1. (a) no wave, (b) longitudinal, (c) longitudinal, (d) either transverse or longitudinal

2. (c) , 3. Sound requires medium for propagation. , 4. (d); 5. (d); 6. 160 m/s;

7. (a) $f = \frac{1}{T} = \frac{1}{4} = 0.25\ Hz$

(b) $v = f\lambda = (0.25\ Hz)(30\ m) = 7.5\ m/s$

(c) The amplitude is half the total range, so A = 1 m.

(d) As each wave passes, the water particles at the surface move in circular orbits of radius r = A = 1 m (see Figure 9). The circumference of such an orbit is $s = 2\pi r = 2\pi(1\ m) = 6.28\ m$. The waves have time period of 4 s, which means that each surface water particle must move through its 6.28 m orbit in 4 s. The velocity of such a water particle is, therefore,

$$v_p = \frac{s}{T} = \frac{6.28\ m}{4\ s} = 1.57\ m/s$$

Note that the wave velocity here is nearly five times greater than the water particle velocity. This signifies that the motion of a wave can be much faster than the motions of the individual particles of the medium in which the wave travels.

8. 0.01 s; 9. 1.66 cm

10. The wave is travelling in the negative x-direction and the wave equation is given as

$$y(x,t) = A\sin\left(\frac{2\pi x}{\lambda} + \frac{2\pi t}{T}\right)$$

Comparing with the above wave equation, the amplitude A = 5. Comparing the second term inside the sine term in the above equation, we get

$$\frac{2\pi}{T} = \pi \Rightarrow T = 2$$

11. (c)

6.6 Waves on a Stretched String:

Consider a uniform stretched string, having mass per unit length m. Under equilibrium conditions, it can be considered to be straight. The x-axis is chosen along the length of the stretched string in its equilibrium state. Let the string be displaced perpendicular to its length by a small amount so that a small section of length Δx is displaced through a distance y from its mean position, as shown in Figure 12. When the string is released, it results in wave motion. Let's see how.

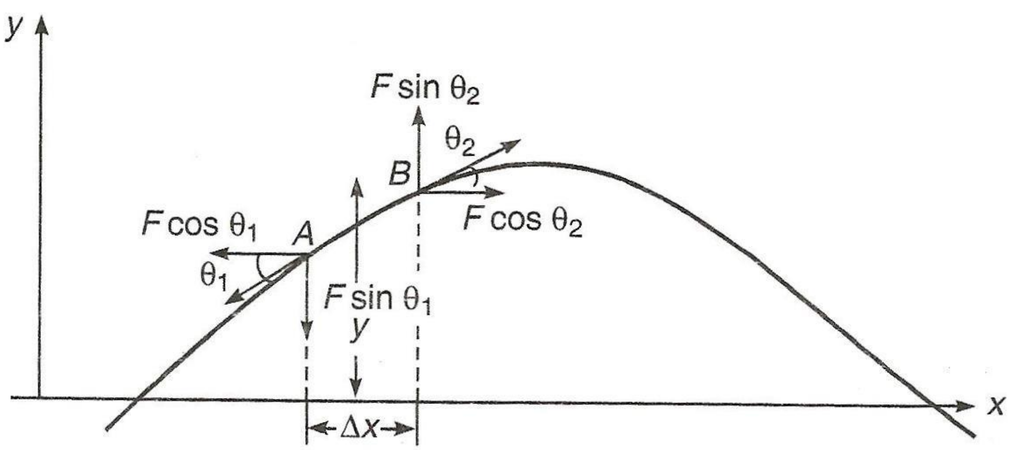


Figure 12: Forces acting on a small element of a string displaced perpendicular to its length.

We have studied that the wave disturbance travels from one particle to another due to their masses (or inertia) and the factor responsible for the periodic motion of the particle is the elasticity of the medium. For a stretched string, the elasticity is measured by the tension F in it and the inertia is measured by mass per unit length or linear mass density, m.

Suppose that the tangential force on each end of a small element AB, as shown in Figure 1, is F; the force on the end B is produced by the pull of the string to the right and the one at A is due to the pull of the string to the left. Due to the curvature of the element AB, the forces are not directly opposite to each other. Instead, they make angles θ_1 and θ_2 with the x-axis. This means that the forces pulling the element AB at opposite ends, though of equal magnitude, do not exactly cancel each other. In order to calculate the net force along the x- and y-axes, the forces are resolved into rectangular components. The net force in the x and the y directions are respectively given by

$$F_x = F\cos\theta_2 - F\cos\theta_1$$

$$and \quad F_y = F\sin\theta_2 - F\sin\theta_1$$

For small angle approximation, $\cos\theta \approx 1$ and $\sin\theta \approx \theta \approx \tan\theta$. This implies that if the displacement of the string perpendicular to its length is relatively small, the angles θ_1 and θ_2 will be small and there is no net force in the x-direction, and the element AB is only subjected to a net upward force F_y. Under the action of this force, the string element will move up and down. Therefore, the y-component of the force on element AB can be written as

$$F_y = F\tan\theta_2 - F\tan\theta_1$$

We know that the tangent of an angle actually defines the slope at that point. In other words, the tangent define the derivative dy/dx. Using this result, the y-component of force on the element can be approximated as

$$F_y = F\left(\left.\frac{dy}{dx}\right|_{x+\Delta x} - \left.\frac{dy}{dx}\right|_{x}\right) \tag{15}$$

Note that the perpendicular displacement y(x, t) of the string is both a function of the position x and time t. However, equation (15) is valid at a particular instant of time. Therefore, the derivative in this expression should be taken by keeping the time fixed. Therefore, equation (15) can be rewritten as

$$F_y = F\left(\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_{x}\right) \tag{16}$$

For the sake of convenience, let us put

$$f(x) = \left.\frac{\partial y}{\partial x}\right|_{x} \qquad and \qquad f(x+\Delta x) = \left.\frac{\partial y}{\partial x}\right|_{x+\Delta x}$$

in equation (16). Thus, equation (16) becomes

$$F_y = F[f(x+\Delta x) - f(x)] \tag{17}$$

To simplify the above expression, we make use of Taylor series expansion of the function $f(x+\Delta x)$ about the point x:

$$f(x+\Delta x) = f(x) + \left.\frac{\partial f}{\partial x}\right|_{x}\Delta x + \left.\frac{1}{2}\frac{\partial^2 f}{\partial x^2}\right|_{x}\Delta x^2 + \cdots$$

Since, Δx is small , we can ignore the second and the higher order terms in Δx to obtain,

$$f(x+\Delta x) = f(x) + \left.\frac{\partial f}{\partial x}\right|_{x}\Delta x$$

$$= f(x) + \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right)\Delta x$$

$$\Rightarrow f(x + \Delta x) - f(x) = \frac{\partial^2 y}{\partial x^2}\Delta x$$

Inserting the above result in equation (10.3), we get

$$F_y = F\frac{\partial^2 y}{\partial x^2}\Delta x$$

This equation gives the net force on the element AB. We use Newton's second law of motion to obtain the equation of motion of this element, by equating this force to the product of mass and acceleration of the element AB. The mass of the element AB is $m\,\Delta x$. Therefore, we can write

$$m\,\Delta x\frac{\partial^2 y}{\partial t^2} = F\frac{\partial^2 y}{\partial x^2}\Delta x$$

$$\Rightarrow \frac{\partial^2 y}{\partial x^2} = \frac{m}{F}\frac{\partial^2 y}{\partial t^2} \tag{18}$$

Note that even though equation (18) has been obtained for a small element AB, it can be applied to the entire string, since there is nothing special about this particular element of the string. In other words, equation (18) can be applied to all the elements of the string.

Now, let us go back to the sinusoidal wave propagating on the string described by the equation

$$y(x,t) = A\sin(\omega t - kx)$$

If this mathematical form is consistent with equation (18), then we can be sure that such a wave can indeed move on the string. To check this, we calculate the spatial and the temporal partial derivatives of particle displacement y(x, t):

$$\frac{\partial^2 y}{\partial x^2} = -k^2 A\sin(\omega t - kx)$$

$$and \qquad \frac{\partial^2 y}{\partial t^2} = -\omega^2 A\sin(\omega t - kx)$$

Substituting these partial derivatives in equation (18 10.4), we get

$$-k^2 A\sin(\omega t - kx) = \frac{m}{F}[-\omega^2 A\sin(\omega t - kx)]$$

$$\Rightarrow \frac{F}{m} = \left(\frac{\omega}{k}\right)^2 \qquad (19)$$

But, we know that ω/k is the wave speed v, therefore, from the above relation, we get

$$v = \frac{\omega}{k} = \sqrt{\frac{F}{m}} \qquad (20)$$

The above relation tells us that velocity of a transverse wave on a stretched string depends on tension and mass per unit length of the string. Using equation (20), we can write equation (18) as

$$\Rightarrow \frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \qquad (21)$$

This result expresses one-dimensional wave equation. It holds as long as we deal with small amplitude waves. Elasticity provides the restoring force and the inertia determines the response of the medium.

6.7 Longitudinal Waves in a Uniform Rod

Consider a cylindrical metal rod of uniform cross-sectional area. When the rod is struck with a hammer at one end, the disturbance will propagate along it with a speed determined by its physical properties. For simplicity, we assume that the rod is fixed at the left end as shown in Figure 13.

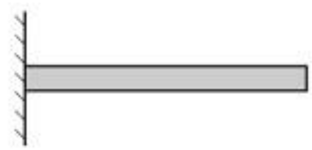


Figure 13: Uniform cylindrical rod fixed at left end.

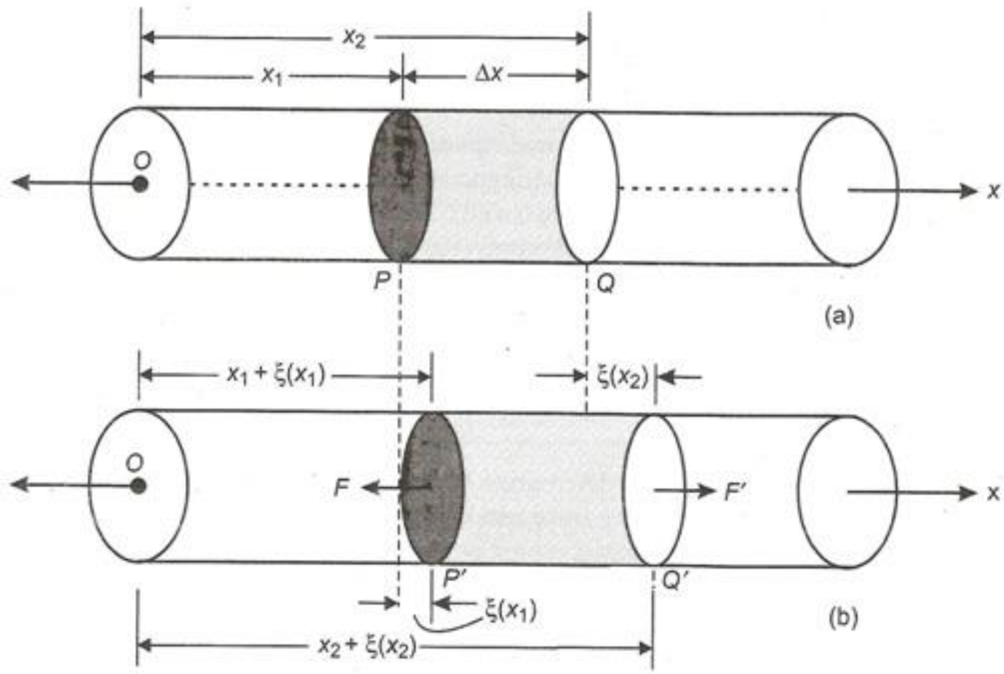


Figure 14: Longitudinal wave propagating in a uniform cylindrical rod. Element PQ in (a) equilibrium state, and (b) deformed state.

We choose x-axis along the length of the rod with origin O at the left end. We divide the rod in a large number of small elements, each of length Δx. Let us consider one such element PQ, as shown in Figure 3a. Since, the rod has been struck at end O lengthwise, the section at P, which is at a distance x_1 from O, will be displaced along x-axis. Since, the force experienced by different sections of the rod is a function of distance, the displacements of particles in different sections will also be function of position. Let us denote is by ξ(x).

Figure 3b shows the deformed state of the rod and displaced position of the element under consideration. Let us denote the x-coordinate of the element in the displaced position by $x_1 + \xi(x_1)$ so that $\xi(x_1)$ represents the displacement of the particles in the section P. Similarly, the new x-coordinate of the particles initially located in the section at Q (x = x_2) be denoted by $x_2 + \xi(x_2)$, so that $\xi(x_2)$ signifies the displacement of the particles in section at Q. Hence, the change in length of the element is $\xi(x_2) - \xi(x_1)$. Using Taylor series expansion of $\xi(x_2)$ around x_1 and retaining the first order terms, just like we did in the case of the string, we can write

$$\xi(x_2) - \xi(x_1) = \left(\frac{\partial \xi}{\partial x}\right)_{x=x_1} \Delta x$$

The linear strain produced in the element PQ can be expressed as

$$\varepsilon(x_2) = \frac{Change\ in\ length}{Original\ length} = \frac{\left(\frac{\partial \xi}{\partial x}\right)_{x=x_1} \Delta x}{\Delta x}$$

$$\Rightarrow \varepsilon(x_2) = \left(\frac{\partial \xi}{\partial x}\right)_{x=x_1} \tag{22}$$

The net force $F' - F$ on the element P'Q' at points P' and Q', as shown in Figure 3b, is toward right. Due to this force, the element under consideration will experience stress, which is the restoring force per unit area. You may recall that the ratio of stress to longitudinal strain defines the Young's modulus Y,

$$Y = \frac{Stress}{Strain}$$

$$\Rightarrow Stress = Y \times Strain$$

In view of the spatial variation of force, we can say that the sections P and Q of the element under consideration will develop different stresses. Therefore, we can write

$$\sigma(x_1) = Y\left(\frac{\partial \xi}{\partial x}\right)_{x=x_1}$$

$$and \qquad \sigma(x_2) = Y\left(\frac{\partial \xi}{\partial x}\right)_{x=x_2}$$

The net stress on the element PQ is

$$\sigma(x_2) - \sigma(x_1) = Y\left[\left(\frac{\partial \xi}{\partial x}\right)_{x=x_2} - \left(\frac{\partial \xi}{\partial x}\right)_{x=x_1}\right]$$

$$= Y[f(x_2) - f(x_1)]$$

where we have put $f(x) = \partial\xi/\partial$x.. As before, using Taylor series expansion for $f(x_2)$ about x_1, we can easily see

$$\sigma(x_2) - \sigma(x_1) = Y\left(\frac{\partial \mathrm{f}}{\partial x}\right)\Delta x$$

$$= Y\frac{\partial}{\partial x}\left(\frac{\partial \xi}{\partial x}\right)\Delta x$$

$$\Rightarrow \sigma(x_2) - \sigma(x_1) = Y\left(\frac{\partial^2 \xi}{\partial x^2}\right)\Delta x \tag{23}$$

If the cross-sectional area of the rod is A, the net force on the elements in the x-direction is given by

$$F(x_2) - F(x_1) = A[\sigma(x_2) - \sigma(x_1)]$$

$$\Rightarrow F(x_2) - F(x_1) = Y\left(\frac{\partial^2 \xi}{\partial x^2}\right)\Delta x \qquad (24)$$

Under dynamic equilibrium condition, the equation of motion of the element PQ, using Newton's second law of motion, can be written as

$$Y\left(\frac{\partial^2 \xi}{\partial x^2}\right)\Delta x = \rho A \Delta x\left(\frac{\partial^2 \xi}{\partial t^2}\right) \qquad (25)$$

where ρ is the density of the material of the rod and $\rho A \Delta x$ signifies the mass of the element PQ. On simplification, we find that the displacement $\xi(x, t)$ satisfies the equation

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{Y}{\rho}\frac{\partial^2 \xi}{\partial x^2} \qquad (26)$$

which is of the form of wave equation (21 10.7) with

$$v = \sqrt{\frac{Y}{\rho}} \qquad (27)$$

Equations (26) and (27) show that the deformation propagates along the rod as a wave and the velocity of the longitudinal waves is independent of the cross-sectional area of the rod.

6.8 Longitudinal Waves in a Gas

Since a gaseous medium lacks rigidity, transverse waves cannot propagate in it; only solids can sustain transverse waves. However, longitudinal waves can propagate in all media such as gas, solid and liquid in the form of compressions and rarefactions. We now discuss longitudinal waves in a gaseous medium.

Sound waves in air columns perhaps are the most familiar one-dimensional waves in a gas. These can be easily excited by placing a vibrating tuning fork at the open end of an air column. What is the basic difference between longitudinal waves in a solid rod that we studied in the last section and a gas column? We know that gases being compressible, the pressure variations in a gas are accompanied by fluctuations in the density, while the density of a solid rod remains essentially constant.

In order to understand the propagation of one-dimensional longitudinal waves in a gas, consider a gas column in a long pipe or cylindrical tube of uniform cross-sectional area A. As before, we conveniently choose x-axis along the length of the tube and divide the column of the gas into small elements or slices, each of small length Δx. Figure 15 shows one such volume element PQRS. Thus, the volume of this element is V = AΔx.

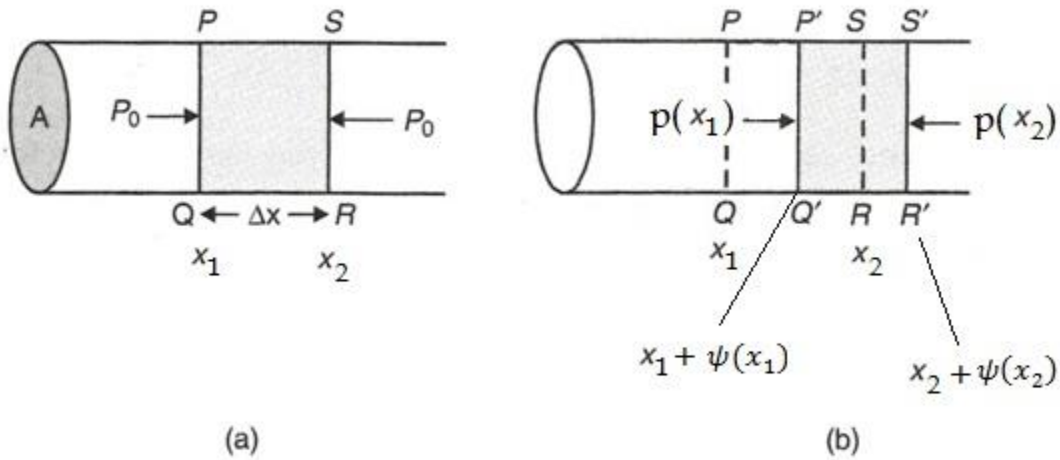


Figure 1: (a) Equilibrium state of the column PQRS of a gas contained in a long tube of cross-sectional area A, and (b) displaced position of column under pressure difference.

Under equilibrium condition, pressure and density of the gas remains the same throughout the volume of the gas, independent of the x-coordinate. Let the equilibrium pressure be denoted by p_0. If the pressure of the gas in the tube is changed, the volume element PQRS will be set in motion giving rise to a net force. Let us choose the origin of the coordinate system so that the particles in plane PQ are at a distance x_1 and those in plane SR are at a distance x_2 from it. Figure 4b shows the displaced position of the volume element when PQ is shifted to P'Q' and SR is shifted to S'R' Let the new coordinates be denoted by $x_1 + \psi(x_1)$ and $x_2 + \psi(x_2)$, respectively. It means that $\psi(x_1)$ and $\psi(x_2)$ respectively, denote the displacements of the particles originally at x_1 and x_2. Therefore, the change in thickness Δl is given by

$$\Delta l = \psi(x_2) - \psi(x_1)$$

If Δl is positive, there is increase in length, and hence the volume of the element also increases and vice versa. Using Taylor series expansion for $\psi(x_2)$ about $\psi(x_1)$, we can write

$$\Delta l = \psi(x_2) - \psi(x_1) = \left(\frac{\partial \psi}{\partial x}\right) \Delta x$$

This means that the change in volume ΔV is

$$\Delta V = A \Delta l = A \Delta x \left(\frac{\partial \psi}{\partial x}\right)$$

The volume strain, which is defined as the change in volume per unit volume, is given by

$$\frac{\Delta V}{V} = \frac{A\Delta x}{A\Delta x}\left(\frac{\partial \psi}{\partial x}\right) = \frac{\partial \psi}{\partial x} \tag{28}$$

This increase in volume of the element is due to the decrease in pressure and vice versa.

It should be noted that until now all the steps that have been followed are identical to the case of the solid rod. However, as mentioned earlier, due to comparatively large compressibility of the gas, change in volume is accompanied by changes in density. This implies that the pressure in the compressed/rarefied gas varies with distance. To proceed further, let us suppose that the pressure at P'Q' is $p_0 + p(x_1')$. Hence, the pressure difference across the ends of the element P'Q'R'S' can be expressed in terms of the pressure gradient,

$$p(x_2') - p(x_1') = \left(\frac{\partial p(x)}{\partial x}\right)_{x=x_1'} \Delta x$$

$$= \frac{\partial(p_0 - \Delta p)}{\partial x}\Delta x$$

Since, p_0 is a constant

$$\Rightarrow p(x_2') - p(x_1') = -\frac{\partial(\Delta p)}{\partial x}\Delta x \tag{29}$$

To express the above result in a familiar form, we note that Δp is connected to the bulk modulus of elasticity by the relation

$$E = \frac{Stress}{Volume\ Strain} = -\frac{\Delta p}{\Delta V/V}$$

The negative sign is included to account for the fact that when the pressure increases, the volume decreases. This ensures that E is positive. We can write the above relation as

$$\Delta p = -E\left(\frac{\Delta V}{V}\right)$$

On substituting for $\Delta V/V$ from equation (28 10.14), we get

$$\Delta p = -E\left(\frac{\partial \psi}{\partial x}\right)$$

Using equation (29 10.15), we find that the pressure difference at the ends of the displaced column is given by

$$p(x_2') - p(x_1') = -\frac{\partial}{\partial x}\left(-E\frac{\partial \psi}{\partial x}\right)\Delta x = E\left(\frac{\partial^2 \psi}{\partial x^2}\right)\Delta x$$

The net force acting on the volume element is obtained by multiplying this expression for pressure difference by the cross-sectional are of the column,

$$F = [p(x_2') - p(x_1')]A$$

$$= EA\Delta x\left(\frac{\partial^2 \psi}{\partial x^2}\right)$$

Under the action of this force, the volume element under consideration shall be set in motion. Using Newton's second law of motion, we find that the equation of motion of the element under consideration can be expressed as

$$\rho\Delta x A\frac{\partial^2 \psi}{\partial t^2} = EA\Delta x\left(\frac{\partial^2 \psi}{\partial x^2}\right)$$

$$\Rightarrow \frac{\partial^2 \psi}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 \psi}{\partial x^2} \tag{30}$$

If we identify the speed of the longitudinal wave as

$$v = \sqrt{\frac{E}{\rho}} \tag{31}$$

equation (30) becomes identical to equation (21). One must note that the wave speed is determined only by the bulk modulus of elasticity and density – two properties of the medium through which the wave is propagating.

When a longitudinal wave propagates through a gaseous medium such as air, the volume elasticity is influenced by the thermodynamic changes that take place in it. These changes can be isothermal or adiabatic. Newton gave the first theoretical expression of the velocity of sound wave in a gas. He assumed that when sound wave travels through a gaseous medium, the temperature variations in the regions of compression and rarefaction are negligible. For sound waves propagating in air, Newton assumed that isothermal changes take place in the medium. For an isothermal change, the volume elasticity equals atmospheric pressure,

$$E = E_T = p$$

Then we can write,

$$v = \sqrt{\frac{p}{\rho}} \tag{32}$$

This is known as the Newton's formula for velocity of sound. For air at STP, $\rho = 1.29\ \mathrm{kgm^{-3}}$ and $\mathrm{p} = 1.01 \times 10^5\ \mathrm{Nm^{-2}}$. Hence, velocity of sound in air at STP, using the Newton's formula comes out to be

$$v = \sqrt{\frac{1.01 \times 10^5\ Nm^{-2}}{1.29\ kgm^{-3}}} = 280\ m/s$$

But experimental results paint a different picture and show that the speed of sound in air at STP is actually around 332 m/s, which is about 15% higher than the value predicted by Newton's formula. This implies that something was wrong with the assumption of isothermal change.

The discrepancy was resolved when Laplace pointed out that sound waves produced adiabatic changes; the regions of compression are hotter while the regions of rarefaction are cooler, i.e. local changes in temperature occur when sound propagates in air. Since, the thermal conductivity of a gas is small and these thermal change occur so rapidly that the heat developed in compression and cooling produced in rarefaction is not transferred out during the short time-scale. The time-scale is the time required by sound to travel from compression to rarefaction. However, the total energy of the system is conserved. This means that the adiabatic changes occur in air when sound propagates.

For an adiabatic change, E_s is γ times the pressure, where γ is the ratio of specific heat capacities of a gas at constant pressure and at constant volume, i.e.

$$E_s = \gamma \mathrm{p}$$

Then, equation (32 10.18) becomes

$$v = \sqrt{\frac{\gamma p}{\rho}} \tag{33}$$

This is known as the Laplace's formula. For air, γ = 1.4 and the velocity of sound in air at STP based on equation (33 10.19) comes out to be 331 m/s, which is in close agreement with the experimentally measured value, thereby establishing the correctness of Laplace's explanation.

At a given temperature, p/ρ is constant for a gas. So, equation (33 10.19) shows that the velocity of a longitudinal wave is independent of pressure.

The question arises that why is the thermal energy unable to flow from a compression to a rarefaction and equalize the temperature creating isothermal conditions? To answer this question, we notice that to attain this condition, thermal energy must flow through a distance of one-half wavelength in a time much shorter than one-half of the period of oscillation of the particles. Thermodynamically, this means that we would need,

$$v_{sound} \ll v_{thermal}$$

One may recall from kinetic theory of gases that the root mean square speed of air molecules is given by

$$v_{rms} = \sqrt{\frac{2k_B T}{m}} \tag{34}$$

where m is the mass of air molecules and T is the absolute temperature in K. We can similarly write the expression for the speed of sound

$$v_{sound} = \sqrt{\frac{\gamma k_B T}{m}} \tag{35}$$

As liquids, in general, are incompressible, the speed of sound in liquids must be significantly higher than in gases. For example, in water whose $E = 2.22 \times 10^9\ Nm^{-2}$, using equation (31) the wave speed comes out to be about 1500 m/s. Even though water is about 1000 times denser than air, sound propagates faster in water than air.

Example 1: Transverse waves are generated in two uniform steel wires A and B of diameters 0.001 m and 0.0005 m, respectively, by attaching their free end to a vibrating source of frequency 500 Hz. Find the ratio of the wavelengths if they are stretched with the same tension.

Solution: The density ρ of a wire of mass M, length L and diameter d is given by

$$\rho = \frac{M}{L\left(\frac{\pi d^2}{4}\right)} = \frac{m}{\left(\frac{\pi d^2}{4}\right)}$$

where m is the linear mass density (mass per unit length). Now, we know that the velocity of a transverse wave in a stretched wire is given by

$$v = \sqrt{\frac{F}{m}}$$

Since, the tension is the same for both the steel wires A and B, therefore, we have

$$\frac{v_A}{v_B} = \sqrt{\frac{m_B}{m_A}}$$

$$\Rightarrow \frac{v_A}{v_B} = \frac{d_B}{d_A}$$

Since, both the wires are made of steel, and have the same densities. Also, we know that the wave velocity v = fλ , where f is the frequency of the source, therefore the above relation can be written as

$$\frac{\lambda_A}{\lambda_B} = \frac{d_B}{d_A} = \frac{0.0005}{0.001} = \frac{1}{2}$$

Self Assessment Question (SAQ) 12: Using dimensional analysis, show that the wave speed v is given by

$$v = K\sqrt{\frac{F}{m}}$$

where K is a dimensionless constant, F is the tension in the string and m is the linear mass density of the string.

Self Assessment Question (SAQ) 13: A one meter long string weighing one gram is stretched with a force of 10 N. Calculate the speed of transverse wave.

Self Assessment Question (SAQ) 14: For a steel rod, $Y = 2 \times 10^{11}\ Nm^{-2}$ and ρ = 7800 kg/m^3. Calculate the speed of the longitudinal waves.

Self Assessment Question (SAQ) 15: In a laboratory experiment (room temperature being 15^{0}C) the wavelength of a note of sound of frequency 500 Hz is found to be 0.68 m. If the density of air at STP is 1.29 kg/m^3, calculate the ratio of the specific heats of air.

Self Assessment Question (SAQ) 16: Write the expressions for speed of mechanical waves in

(a) a string, (b) rod, (c) liquid, (d) gas

Self Assessment Question (SAQ) 17: Explain why the velocity of sound is generally greater in liquids than in gases.

Self Assessment Question (SAQ) 18: Choose the correct option.

The speed of sound in air is

(a) $\propto \sqrt{pressure\ of\ air}$
(b) $\propto pressure\ of\ air$
(c) $\propto square\ of\ pressure\ of\ air$

(d) $independent\ of\ pressure\ of\ air$

Self Assessment Question (SAQ) 19: Choose the correct option:

Velocity of sound is measured in hydrogen and oxygen gases at a given temperature. The ratio of the two velocities will be:

(a) 1:4, (b) 4:1, (c) 1:1, (d) 32:1

Answer of Selected Self Assessment Questions (SAQs):

12. We have to check if the following dimensional formula for wave speed is correct or not, i.e. it has units of velocity or not.

$$v = K\sqrt{\frac{F}{m}}$$

- K is the dimensionless constant.
- F has units of N or kg m/s^2. Therefore,

$$[F] = \frac{[M][L]}{[T]^2}$$

- m has units of kg/m. Therefore,

$$[m] = \frac{[M]}{[L]}$$

Hence, it can be shown that the formula for the wave speed is dimensionally correct.

$$[v] = \sqrt{\frac{\frac{[M][L]}{[T]^2}}{\frac{[M]}{[L]}}} = \frac{[L]}{[T]}$$

13. We know that the velocity of a transverse wave on a stretched string is related to tension and mass per unit length of the string by the following relation

$$v = \sqrt{\frac{F}{m}} = \sqrt{\frac{10\ N}{0.001\ kg/m}}$$

$$\Rightarrow v = 100\ m/s$$

14. We know that the velocity of a longitudinal wave in a uniform solid rod is related to the Young's modulus of the material of the rod and its density by the following relation

$$v = \sqrt{\frac{Y}{\rho}} = \sqrt{\frac{2 \times 10^{11}\, Nm^{-2}}{7800\, kg/m^3}}$$

$$\Rightarrow v = 5060\, m/s$$

15. Velocity of sound at 15^0C = fλ = 500 X 0.68 = 340 m/s.

Velocity of sound at 0^0C is given as

$$= (340\, m/s)\sqrt{\frac{273\, K}{(273 + 15)\, K}} = 331\, m/s$$

Since, the velocity of sound in air in a gas is given by

$$v = \sqrt{\frac{\gamma p}{\rho}}$$

$$\Rightarrow \gamma = v^2 \left(\frac{\rho}{p}\right)$$

$$= \left(331 \frac{m}{s}\right)^2 \left(\frac{1.29\, kg/m^3}{1.01 \times 10^5\, Pa}\right) = 1.39$$

16. (a) $\sqrt{\frac{F}{m}}$ (b) $\sqrt{\frac{Y}{\rho}}$ (c) $\sqrt{\frac{E}{\rho}}$, (d) $\sqrt{\frac{\gamma p}{\rho}}$; 17. Refer to the text; 18. (d); 19. (b)

Example 2: A uniform rope of length 12 m and mass 6 kg hangs vertically from a rigid support. A block of mass 2 kg is attached to the free end of the rope. A transverse pulse of wavelength 0.06 m is produced at the lower end of the rope. What is the wavelength of the pulse when it reaches the top of the rope? [$g = 9.8\, ms^{-2}$]

Solution: As the rope has mass and a mass is also suspended from the lower end, the tension in the rope will be different at different points. We know that the speed of a transverse wave is given by

$$\mathrm{v} = \sqrt{\frac{\mathrm{F}}{\mathrm{m}}}$$

Therefore, the ratio of speeds of transverse wave at the top and at the bottom of the rope is

$$\frac{v_T}{v_B} = \sqrt{\frac{F_T}{F_B}} = \sqrt{\frac{(6+2) \times 9.8 \text{ N}}{2 \times 9.8 \text{ N}}}$$

$$\Rightarrow \frac{v_T}{v_B} = 2$$

Since, the frequency is the characteristic of the source producing the waves, we have frequency of the wave at the top is the same as the frequency at the bottom of the rope. Since, wave speed $v = f\lambda$, we have

$$\lambda_T = 2\lambda_B = 2 \times 0.06 = 0.12 \text{ m}$$

6.9 SUMMARY

In this unit, we have studied about the different waves that are familiar to us and are part of our everyday life. Then we studied what is meant by the wave motion, the formation and propagation of waves in a medium. We learned about the difference between transverse and longitudinal waves, and how to represent a wave at a fixed position and at a fixed time graphically. We wrote the mathematical expression of a progressive wave corresponding to a given set of wave parameters and travelling along +x /–x directions.

We defined the terms that are needed to describe a wave such as amplitude, time period, wavelength, frequency, wave number, angular frequency, wave velocity etc and understood how they are related to each other. Finally, we derived the relationship between the velocity of a particle in the medium and the velocity of the wave at any instant.

6.10 GLOSSARY

Amplitude – the maximum displacement of a wave from equilibrium (e.g. height of a transverse wave from the middle).

Displacement – net change in location of a moving body. It is measured from the equilibrium position.

Elasticity – ability of a material to regain its shape after being distorted.

Force – any interaction that, when unopposed, can change the state of motion of an object.

Frequency – the number of complete cycles per second made by a wave. The SI unit of frequency is the hertz (Hz), which is equal to 1 cycle per second.

Longitudinal waves – waves in which the vibrations are parallel to the direction of travel of the wave.

Microwaves – electromagnetic waves of wavelength between about 0.1 mm and 10 mm.

Molecule – the smallest amount of a compound or element that can exist independently.

Momentum – mass multiplied by velocity.

Pressure – force per unit area applied at right angles to a surface. The SI unit of pressure is the pascal (Pa), which is equal to 1 N/m^2.

Radio waves – electromagnetic waves of wavelength longer than about a millimeter.

Sound – vibrations in a substance that travel through the substance.

Speed – the ratio of distance traveled and time. The SI unit of speed is m/s.

Transverse waves – waves in which the vibrations are at right angles to the direction of propagation of wave.

Ultraviolet radiation – electromagnetic waves between the violet end of the visible spectrum (wavelength ~400 nm) and X-rays (wavelength less than ~1 nm).

Velocity – speed in a given direction.

Wavelength – the distance between two adjacent wave-crests.

X-rays – electromagnetic waves of wavelength less than about 1 nm.

6.11 REFERENCES

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2. The Physics of Waves and Oscillations, N K Bajaj – Tata McGraw-Hill, New Delhi
3. Fundamentals of Physics, David Halliday, Robert Resnick, Jearl Walker – John Wiley & Sons
4. Physics, Jim Breithaupt – Palgrave
5. Applied Physics, Arthur Beiser – McGraw-Hill Company

6.12 SUGGESTED READINGS

1. Fundamentals of Physics, David Halliday, Robert Resnick, Jearl Walker – John Wiley & Sons
2. Berkeley Physics Course Vol 3, Waves, C Kittel et al, McGraw- Hill Company

6.13 TERMINAL QUESTIONS

1. A tuning fork vibrating at 300 Hz is placed in a tank of water. (a) Find the frequency and wavelength of the sound waves in the water. (b) Find the frequency and wavelength of the sound waves produced in the air above the tank by the vibrations of the water surface. The velocity of the sound is 4913 ft/s in water and 1125 ft/s in air.

2. The visible region of the electromagnetic spectrum begins from 400 nm. Calculate the corresponding frequency.

3. The equation for the displacement of a stretched string is given by

$$y = 4\sin 2\pi\left[\frac{t}{0.02} - \frac{x}{100}\right]$$

where y and x are in cm and t is in seconds. Determine the

(a) direction in which the wave is propagating
(b) amplitude
(c) time period
(d) frequency
(e) angular frequency
(f) wavelength
(g) velocity of wave
(h) wave number

4. What quantity is carried off by all types of waves from their source to the place where they are eventually absorbed?

5. A wave of frequency f_1 and wavelength λ_1 goes from a medium in which its velocity is v to another medium in which its velocity is 2v. Find the frequency and wavelength of the wave in the second medium.

6. A violin string is vibrating at a frequency of 440 Hz. How many vibrations does the string make while its sound travels 200 m in air?

7. Lower the frequency of a wave

(a) higher is its velocity.

(b) longer is its wavelength

(c) smaller is its amplitude

(d) shorter is its period

8. Which of the following is an entirely longitudinal wave?

(a) Water wave

(b) Sound wave

(c) Electromagnetic wave

(d) A wave in a stretched string

9. Sound cannot travel through

(a) vacuum

(b) liquid

(c) gas

(d) solid

10. Of the following properties of a wave, the one that is independent of the others is

(a) velocity

(b) frequency

(c) wavelength

(d) amplitude

11. Write notes on:

(i) Wave Formation and Propagation (ii) Transverse and Longitudinal Waves

(iii) Wave Properties

12. What is meant by wave equation? Derive the wave equation when a wave is travelling in the negative x-direction.

ANSWERS

Selected Terminal Questions:

1. (a) In the water, the frequency of the sound waves is the same as the frequency of their source, and their wavelength is

$$\lambda_1 = \frac{v_1}{f} = \frac{4931\ ft/s}{300\ Hz} = 16.4\ ft$$

(b) In the air, the frequency of the sound waves is the same as the frequency of their source, but the wavelength differs from that in the water

$$\lambda_2 = \frac{v_2}{f} = \frac{1125\ ft/s}{300\ Hz} = 3.75\ ft$$

2. $f = c/\lambda = (3 \times 10^8\ m/s)/(400 \times 10^{-9}\ m)$

$$= 7.5 \times 10^{14}\ Hz$$

3. (a) As there is a negative sign between t and x terms, the wave is propagating along the positive x-axis.

(b) A = 4 cm

(c) T = 0.02 s

(d) f = 1/T = 50 Hz

(e) $\omega = 2\pi f = 100\pi$ rad/s

(f) $\lambda = 100$ cm

(g) $v = f\lambda = 50$ m/s

(h) $k = 2\pi/\lambda = \pi/50\ cm^{-1}$

4. Energy

5. The frequency of the wave remains constant, therefore, $f_2 = f_1$. The wavelength meanwhile will change according to the relation

$$\frac{v_1}{\lambda_1} = \frac{v_2}{\lambda_2}$$

$$\Rightarrow \lambda_2 = \frac{v_2}{v_1}\lambda_1 = 2\lambda_1$$

6. The speed of sound wave (v) in air about 330 m/s. Therefore, the wavelength of the sound wave produced by the violin string will be given as

$$\lambda = \frac{v}{f} = \frac{330\ m/s}{440\ Hz} = 0.75\ m$$

Hence, to travel 200 m, the number of vibrations will be

$$= \frac{200}{0.75} = 266$$

7. (b); 8. (b); 9. (a); 10. (d)

UNIT 7: LISSAJOUS FIGURES

Structure of the Unit

7.1 INTRODUCTION

In this unit, we will discuss the superposition of two harmonic oscillations that are orthogonally perpendicular to one another and we will also study how Lissajous figures can be used to represent the path of the resulting motion. In order to obtain the resultant of two or more harmonic oscillations, we will use a very important principle called the superposition principle which states that, " The resultant of two or more harmonic displacements is simply the algebraic sum of the individual displacements". In this unit, we shall discuss the validity of this principle.

7.2 OBJECTIVES

After studying this unit, the learners will be able to explain about

- The Lissajous figures for the resultant of two mutually perpendicular harmonic oscillations with same frequencies
- The Lissajous figures for the resultant of two mutually perpendicular harmonic oscillations with frequency ratio(1:2)
- Demonstration of Lissajous figures
- Uses of Lissajous figures

7.3 TWO MUTUALLY PERPENDICULAR HARMONIC VIBRATIONS

In general, it's essential to find a way to combine the effect of two oscillations acting on the same body simultaneously. You will learn how to accomplish this situation of two separate SHMs in this part that as follows.

7.3.1 Oscillations Having Same Frequencies

Now we consider a particle moving under the simultaneous influence of two perpendicular harmonic oscillations of equal frequency, one along the x-axis, the other along the y-axis. Let A_1 and A_2 respectively be the amplitudes of the x and y oscillations. For simplicity, let us assume that the phase constant of the x oscillation is zero and that of the y oscillation is δ, so that is the phase difference between them. Thus, the two rectangular SHMs can be written as

$$x = A_1 \cos \omega t \qquad (1)$$

$$y = A_2 \cos(\omega t + \delta) \qquad (2)$$

Here x and y are the displacements along two mutually perpendicular directions. The resultant motion of the particle at any time t can be obtained as discussed below: As this motion is in two dimensional and the resultant path traced by the particle can be obtained by eliminating t from the above equation (1) and (2) which comes out to be an ellipse.

From equation (1), we can write

$$\cos\omega t = \frac{x}{A_1} \text{ ; which gives } \quad \sin\omega t = \sqrt{1 - \left(\frac{x}{A_1}\right)^2}$$

Putting these values in equation (2), we have

$$y = A_2 \cos(\omega t + \delta) = A_2[\cos\omega t \cos\delta - \sin\omega t \sin\delta]$$

$$= A_2 \left[\left(\frac{x}{A_1}\right)\cos\delta - \left(\sqrt{1 - \left(\frac{x}{A_1}\right)^2}\right)\sin\delta\right]$$

Or,

$$\left(\frac{x}{A_1}\cos\delta - \frac{y}{A_2}\right)^2 = \left(1 - \left(\frac{x}{A_1}\right)^2\right)\sin^2\delta$$

By simplifying the above equation one can write it as

$$\therefore \frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} - \frac{2xy\cos\delta}{A_1 A_2} = \sin^2\delta \qquad (3)$$

As we can see, This general equation (3) is an equation of an ellipse, whose axes are incline to the coordinate axes. Thus,we came to conclusion that the resultant motion of the particle is along an elliptical path.

Equation (3) shows that x remains between $-A_1$ and A_1 and that of y remains between $-A_2$ and A_2. Thus, the particle always remains inside the rectangle defined by

$$x = \pm A_1 \quad and \quad y = \pm A_2$$

The ellipse given by equation (3) is shown in the figure below:

Now we shall consider some special cases

(a) The two component SHMs are in phase, $\delta = 0$
(b) The two component SHMs are out of phase, $\delta = \pi$
(c) The phase difference between the two component SHMs, $\delta = \pi/2$
(d) The phase difference between the two component SHMs, $\delta = 3\pi/2$

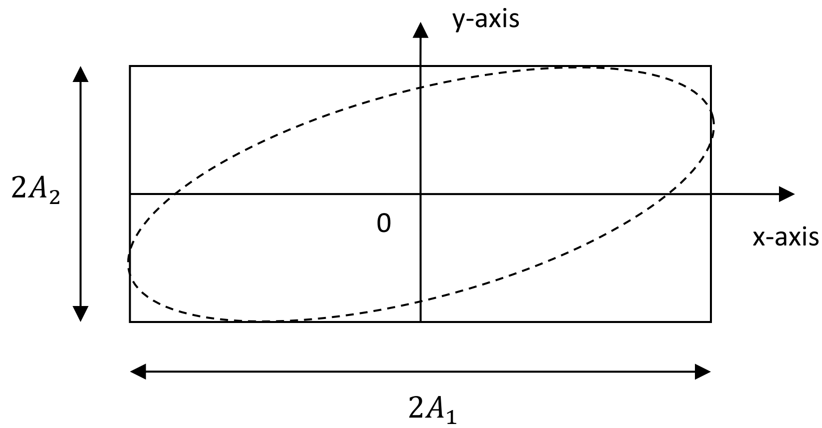

Fig.1: Elliptical path

Let us now discuss the resultant motion of the particle under by taking into the consideration the above special cases.

(a) When the two superposing SHMs are in phase, $\delta = 0$ and equation (3) reduces to

$$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} - \frac{2xy}{A_1 A_2} = 0$$

Or,

$$\left(\frac{y}{A_2} - \frac{x}{A_1}\right)^2 = 0$$

$$\therefore y = \frac{A_2}{A_1} x \tag{4}$$

Equation (4) is an equation of a straight line passing through the origin and having a positive slope of $\frac{A_2}{A_1}$ and passing through the origin.The figure below shows the path followed by the particle in this case. The particle moves on the diagonal (shown by the dotted line) of the rectangle.

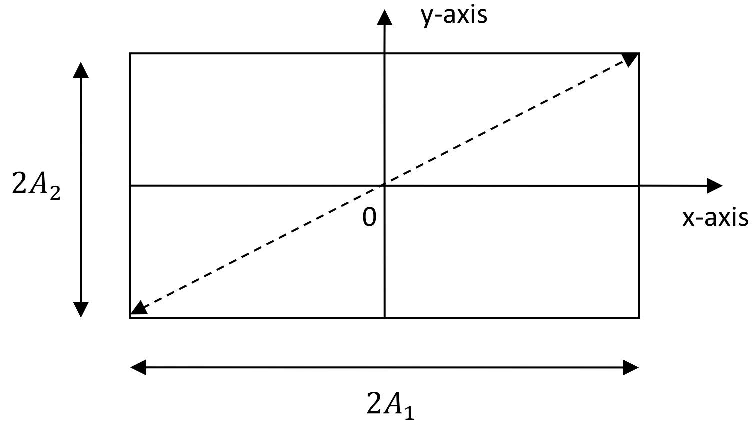

Figure2: Superpostion of two perpendicular SHMs of the same frequency for the phase difference $\boldsymbol{\delta = 0}$.

Equation (4) can also be obtained directly from equations (1) and (2) putting $\delta = 0$.

The displacement of the particle on this straight line at any time t is

$$x = A_1 cos\omega t$$

$$y = A_2 cos\omega t$$

Then one can write in simpler terms

$$r = \sqrt{x^2 + y^2}$$

$$= \sqrt{(A_1 \cos \omega t)^2 + (A_2 \cos \omega t)^2} = \sqrt{A_1^2 + A_2^2} \cos \omega t$$

Thus, we can see that the resultant motion is also SHM with the same frequency and phase as the component motions.

(b) Now we shall consider the case when the two superposing SHMs are out of phase i.e., the phase difference between them is $\delta = \pi$.

Then from equation (3), we can wtite it as

$$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} + \frac{2xy}{A_1 A_2} = 0$$

Or,

$$\left(\frac{y}{A_2} + \frac{x}{A_1}\right)^2 = 0$$

$$\therefore y = -\frac{A_2}{A_1}x \qquad (5)$$

This equation (5) represents a pair of coincident straight lines passing through the origin and having a negative slope $-\frac{A_2}{A_1}$.The figure below shows the path followed by the particle.

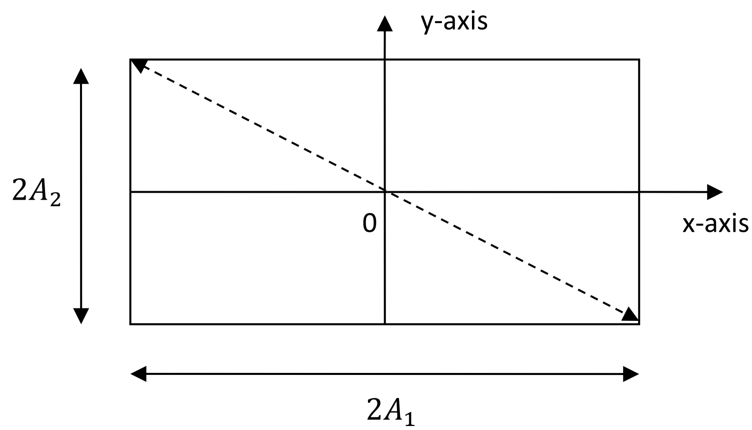

Figure 3: The straight line path traced by the resultant motion of the particle with phase difference, $\boldsymbol{\delta = \pi}$.

Equation (5) can also be obtained directly on the basis of equations (1) and (2) and putting $\delta = \pi$. Further, the displacement of the particle on this straight line path at a given time t is

$$r = \sqrt{x^2 + y^2} = \sqrt{(A_1 \cos\omega t)^2 + (A_2 \cos(\omega t + \pi))^2}$$

$$= \sqrt{(A_1 \cos\omega t)^2 + (-A_2 \cos)^2} = \sqrt{A_1^2 + A_2^2}\cos\omega t$$

Thus, one can see that the resultant motion is also SHM with the same frequency having amplitude of the resultant SHM is $\sqrt{A_1^2 + A_2^2}$.

(c) Now we shall discuss one interesting case, when the phase difference between the two component SHMs is $\delta = \pi/2$.

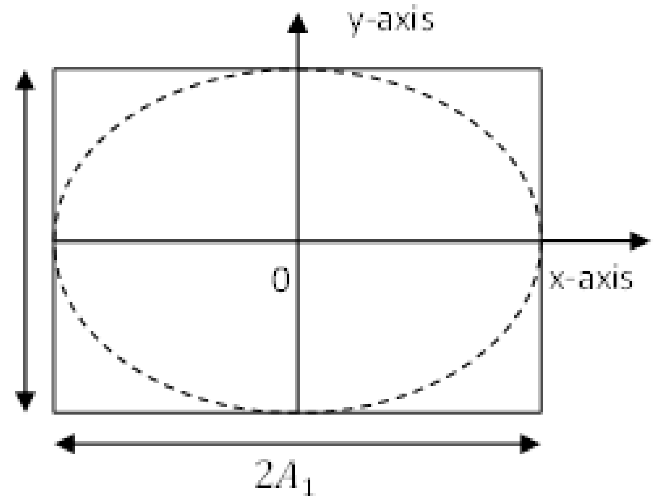

Figure 4: The resultant motion of the particle when the phase difference, $\boldsymbol{\delta = \pi/2}$.

From equation (3), we have

$$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} = 1 \tag{6}$$

The above equation is a standard equation of an ellipse with its axes along the x-axis and the y-axis and with its center at the origin. The lengths of the major and the minor axes are $2A_1$ and $2A_2$, respectively. The path traced by the particle (shown by the dotted line) is depicted in Fig. 4.

In case the amplitudes of the two individual SHMs are equal, $A_1 = A_2 = A$, i.e. the major and the minor axes are equal, then the ellipse reduces to a circle.

$$x^2 + y^2 = A^2 \tag{7}$$

Thus, the resultant motion of a particle due to superposition of two mutually perpendicular SHMs of equal amplitude and having a phase difference of $\pi/2$ is a *circular motion*. The circular motion may be clockwise or anticlockwise depending on which component leads the other.

(d) we shall consider the case when the when the phase difference between the two component SHMs is $\delta = 3\pi/2$.

$$x = A_1 cos\omega t$$

$$y = A_2 \cos(\omega t + \frac{3\pi}{2}) = A_2\ sin\omega t$$

Which gives

$$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} = 1$$

We have an ellipse of the same form as obtained for the case(c) but the motion is now counter-clockwise. In optics such a vibration is called as left-handed elliptically polarized vibrations.

7.3.2 Oscillations Having Different Frequencies (Lissajous Figures)

When the frequencies of the two perpendicular SHMs are not equal, the resulting motion becomes more complicated. The patterns, that are traced by a particle which is subjected simultaneously to two perpendicular SHMs of different frequencies, are known as Lissajous figures, after J.A. Lissajous (1822-1880) who made an extensive study of these motions. We shall also study with few examples to illustrate the shape of the Lissajous figure for some of the special cases.

(I)Analytical Method :-

For this case let us consider that frequency ω_2 of the y oscillation is twice the frequency ω_1 of the oscillation, i.e., $\omega_1 = \omega$ and $\omega_2 = 2\omega$. Then the two SHMs are then given by

$$x = A_1 \cos \omega_1 t \qquad (8)$$

$$y = A_2 \cos(\omega_2 t + \delta) \qquad (9)$$

The phase difference between them at any instant t, is given by

$$\Delta\varphi = (\omega_2 t + \delta) - \omega_1 t$$

$$= (\omega_2 - \omega_1)t + \delta \qquad (10)$$

Given that the superimposed orthogonal oscillations have different frequencies, one of them will change more quickly than the other and move ahead of the other in phase. As a result, there are many stages in the pattern of the resulting motion. It evolves with time as a result of the change in the phase difference, which is similarly a function of time. Though the broad outline of the resulting oscillation is similar to that found for the situation of equal frequencies, i.e. the motion is confined within a rectangle with sides $2A_1$ and $2A_2$.

$$x = A_1 \cos \omega t \qquad (11)$$

$$y = A_2 \cos(2\omega t + \delta) \qquad (12)$$

If we eliminate t from these two equations(11) and (12), we can determine the equation of the trajectory of the particle.

From the equation(11), one can write

$$\cos\omega t = \frac{x}{A_1}$$

And, expanding the equation(12), we have

$$y = A_2 \cos(2\omega t + \delta)$$

$$y = A_2[cos2\omega t\ cos\delta - sin2\omega t\ sin\delta]$$

$$= A_2[(2\cos^2\omega t - 1)\cos\delta - 2\sin\omega t\cos\omega t\sin\delta]$$

Substituting y (x/A_1) for $\cos\omega t$ in the above expression, we get

$$\frac{y}{A_2} = \left\{2\left(\frac{x}{A_1}\right)^2 - 1\right\}\cos\delta - 2\left(\frac{x}{A_1}\right)\sqrt{1-\left(\frac{x}{A_1}\right)^2}\sin\delta$$

After rearranging the terms, we can have the expression as

$$\left(\frac{y}{A_2} + \cos\delta\right) - 2\left(\frac{x}{A_1}\right)^2\cos\delta = -2\left(\frac{x}{A_1}\right)\sqrt{1-\left(\frac{x}{A_1}\right)^2}\sin\delta$$

If we square both side to the above expression simply it , we get

$$\left(\frac{y}{A_2} + \cos\delta\right)^2 + \frac{4x^2}{A_1^2}\left(\frac{x^2}{A_1^2} - 1 - \frac{y}{A_2}\cos\delta\right) = 0 \qquad (13)$$

The above equation is of fourth degree, which, in general, represents a closed curve having two loops. For a given value of δ, the curve corresponding to the above equation(13) can be traced using the knowledge of coordinate geometry.

Now let us discuss the case when$\delta = 0$. Thus, we have$\cos\delta = 1$. The above equation (13) reduces to

$$\left(\frac{y}{A_2} + 1\right)^2 + \frac{4x^2}{A_1^2}\left(\frac{x^2}{A_1^2} - 1 - \frac{y}{A_2}\right) = 0$$

$$\therefore \left(\frac{y}{A_2} + 1 - \frac{2x^2}{A_1^2}\right)^2 = 0$$

This represents two coincident parabolas with their vertices at $(0, -A_2)$ as shown in Fig.6(using dotted lines). The equation of each parabola is being represented as

$$\frac{y}{A_2} + 1 - \frac{2x^2}{A_1^2} = 0$$

$$or \qquad x^2 = \frac{A_1^2}{2A_2}(y + A_2) \qquad (14)$$

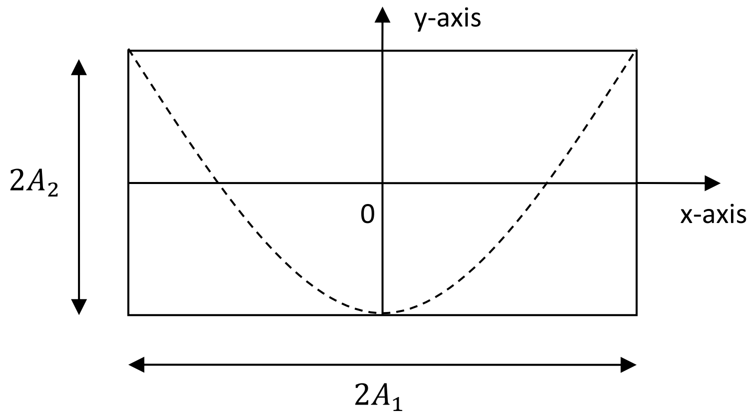

Fig.5: Superposition of two mutually perpendicular SHMs with frequencies in the ratio 1:2 and phase difference equal to zero(**$\delta = 0$**).

Graphical method : -

The analytical method discussed above becomes very cumbersome when δ takes other value than zero. In such cases, the resultant motion can be constructed quite conveniently by using graphical method.

For understanding how the lissajous figures can be formed using graphical method let us discuss with an example as shown below

Example: A particle is subjected to two mutually perpendicular simple harmonic oscillations,

$$x = 2\cos t$$

$$y = \cos(t + 4)$$

Trace the trajectory of the particle using graphical method.

Solving the above case we have set up a table of values(i.e. Table 1) to see what is happening. We give each point a "point number" so that it is easier to understand when we graph the curve.

Table.1

t	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
x	2	1.4	0	-1.4	-2	-1.4	0	1.4	2
y	-0.6	0.1	0.7	1	0.7	-0.1	-0.8	-1	-0.7
Pt.no.	1	2	3	4	5	6	7	8	9

From the above table1, the resulting curve, with the numbered points included, is shown in the following figure. Point 1 is actually equivalent to Point 9.

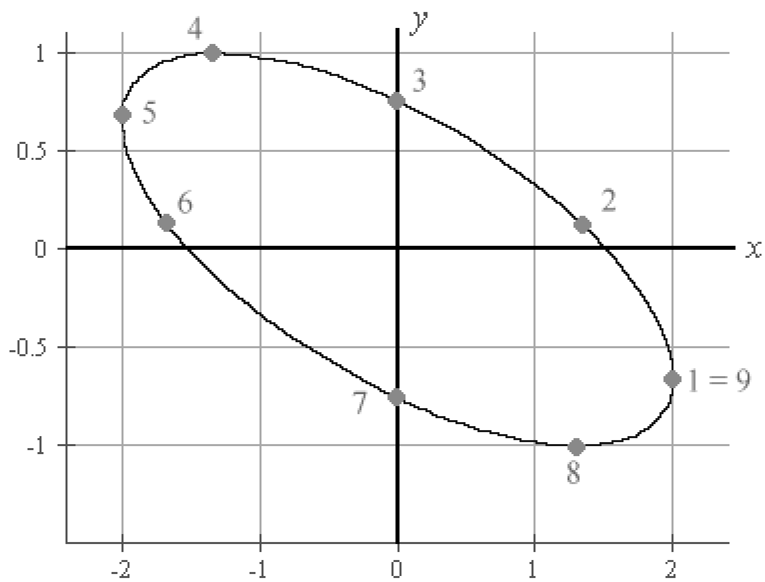

Figure. 6

7.4 DEMONSTRATION OF LISSAJOUS FIGURES

A visual record of Lissajous figures can be obtained be means of a cathode-ray oscillograph (as shown in Fig.8). Here, two rectangular oscillations are simultaneously imposed upon a beam of cathode rays by connecting two sources of electrical oscillations to horizontal plates XX and vertical plates YY of the oscillograph. Thus the beam of cathode rays is subjected simultaneously to two perpendicular deflections. The beam falls on a fluorescent screen on which the Lissajous figure corresponding to the resultant motion can be seen. If the frequencies of the electrical oscillations are not exactly in a simple ratio, the figure will be seen to change its form slowly. For more complicated frequency ratios, very beautiful patterns are obtained in oscillograph.

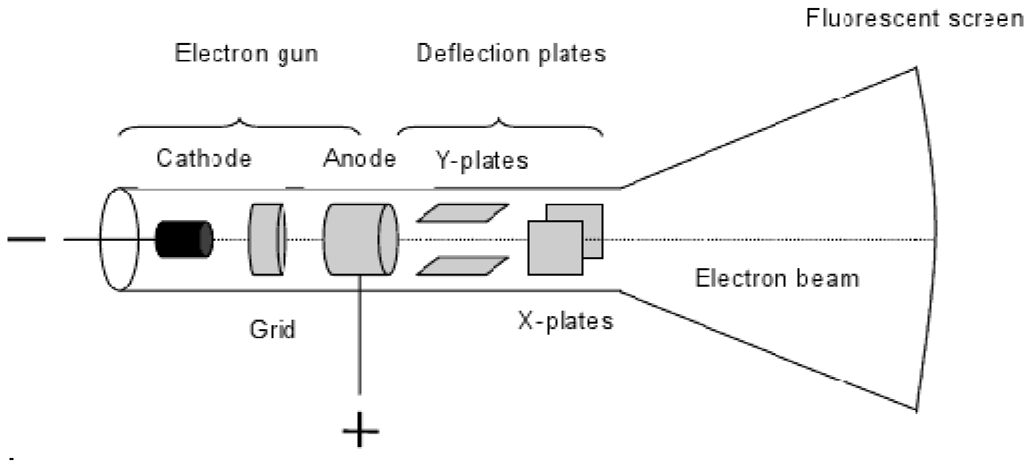

Fig. 8: The Cathode Ray Oscillograph

7.5 USES OF LISSAJOUS FIGURES

Lissajous figures can be used to determine the ratio of two exactly commensurate frequencies. The Lissajous figure is steady and, by inspection, we can find the ratio of the frequencies of the component oscillations. Let ν_1 and ν_2 be the frequencies of the oscillations along x and y axes respectively and let t_c be the time during which a complete cycle of the figures is described. Then, during one cycle, the number of oscillations made by the particle parallel to the x-axis will be $\nu_1 t_c$ and that of the oscillations parallel to the y-axis will be $\nu_2 t_c$.Hence $\frac{\nu_1}{\nu_2} = \frac{\nu_1 t_c}{\nu_2 t_c}$.

In other words, the ratio of the frequencies of the x and y oscillations will be equal to the inverse ratio of the maximum number of intersections of the Lissajous figure on the two lines parallel to the x and y axes respectively.

Lissajous figures may also be used to compare two nearly equal frequencies. If the frequencies of the two component oscillations are not exactly equal, the Lissajous figure will change gradually, as discussed earlier in this unit. We have seen that, if ν_1 and ν_2 are nearly equal frequencies and fe is the time for a complete cycle of change of Lissajous figure

$$\nu_1 - \nu_2 = \pm \frac{1}{t_c}$$

The sign may be determined by observing the direction of change of the pattern to find out which of the two oscillations gains over the other.

7.6 SUMMARY

After studying this unit, the learners have learnt about

- The resultant of two mutually perpendicular harmonic oscillations with same frequencies
- The Lissajous figures for the resultant of two mutually perpendicular harmonic oscillations with frequency ratio(1:2)
- Demonstration of Lissajous figures
- Uses of Lissajous figures

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2. Concepts of Physics, Part I, H C Verma – Bharati Bhawan, Patna
3. The Physics of Waves and Oscillations, N K Bajaj – Tata McGraw-Hill, New Delhi
4. Fundamentals of Physics, David Halliday, Robert Resnick, Jearl Walker – John Wiley & Sons

7.8 SUGGESTED READINGS

1. Waves and Oscillations, R.N.Chaudhuri, New Age International(P)Limited,Publishers.
2. Fundamentals of Physics, David Halliday, Robert Resnick, Jearl Walker – John Wiley & Sons
3. Berkeley Physics Course Vol 3, Waves, C Kittel et al, McGraw- Hill Company

7.9 TERMINAL QUESTIONS

1. A particle is subjected to three simple harmonic oscillations, one along the x-axis, second along the y-axis and the third along the z-axis. The three motions are given by

$$x = A \sin \omega t$$

$$y = B \sin \omega t$$

$$z = C \sin \omega t$$

   Calculate the amplitude of the resultant motion.

2. A body is executing simple harmonic motion, and its displacement at time t is given by

$$x = 5 \sin 3\pi t$$

   Plot the displacement, velocity, and acceleration for two complete periods.

3. A particle is under the influence of two simultaneous SHMs in mutually perpendicular directions given by

$$x = \cos \pi t$$

$$y = \cos \frac{\pi t}{2}$$

determine the trajectory of the resulting motion of the particle.

4. Write short note on Lissajous' figures. How are they demonstrated experimentally?
5. Describe and obtain formulae for the superposition of two mutually perpendicular SHMs with equal frequencies.
6. Determine the the shape of the Lissajous figure for the resultant motion, if a particle is subjected to the following SHMs:

$$x = 2 \sin 2\pi t$$

$$y = 3 \sin \pi t$$

7. A particle is simultaneously subjected to two simple harmonic motions in the same direction in accordance with the following equations:

$$y_1 = 8 \sin 2\pi t \quad and \quad y_2 = 4 \sin 6\pi t$$

Show graphically the resultant path of the particle.

8. Construct the Lissajous figures for the following component oscillations. If you are using graphical method, you may have to take more than 9 points to get the complete graph in some cases.

(a) $x = 2\ \sin t\,, y = \cos 2t$

(b) $x = \sin t\,, y = \cos(t + \pi/4)$

(c) $x = \sin \pi t\,, y = 2 \sin\left(\pi t + \frac{\pi}{2}\right)$

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UNIT 8 CARNOT ENGINE AND ENTROPY

CONTENTS

8.1 INTRODUCTION

The law of conservation of energy serves as the foundation for the first law of thermodynamics. Although it aids in the measurement of changes in various entities, like heat, internal energy, work completed, etc., it is unable to demonstrate the viability of a reaction occurring spontaneously in real life. There are numerous instances of processes that the first law fully permits but which are not practical. For instance, water kept in a bucket won't automatically freeze. If we adhere to the first law of thermodynamics, it is conceivable for a portion of water to absorb heat energy from the remaining water and evaporate. Due to a loss of heat energy, the remaining water will freeze. Practically speaking, though, this is not possible. This leads us to the conclusion that more rules must be established in order to completely understand the process's spontaneity. The second law of thermodynamics puts some limitations on the efficiency of the process which helps in enabling us to determine the process' viability. This law is crucial to understand how refrigerators and heat engines operate. After studying this chapter, you will understand that the second law of thermodynamics suggests a refrigerator's performance coefficient cannot be infinite and a heat engine's efficiency cannot be unity. In the last section of this unit, we will study about the entropy and its physical significance which plays an important role in studying thermodynamics.

8.2 OBJECTIVES

By the end of this unit, you will be able to -

- Explain Basic function of heat Engine
- Explain heat engine and its various parts
- Differentiate between reversible and irreversible processes
- Evaluate efficiency of heat engines
- Learn about Carnot cycle and Carnot engine
- Entropy and its physical significance
- Change in entropy in reversible and irreversible process

8.3 HEAT ENGINE

Any device which converts heat continuously into mechanical work is called a heat engine. This idea of conversion of heat into work has come from very early times when it was observed that bodies when heated develop power. Thus for instance, when water is boiled in a vessel closed by a lid, the steam generated inside throw the lid off, showing thereby that high pressure steam can be made to do work. Similarly the gun powder and other explosives do work in breaking rocks etc. Likewise, the high velocity winds, caused by the heating of parts of earth's surface by the sun, do work in driving wind mills and in propelling ships with the aid of sail etc.

All these simple and elementary facts indicate that transference of heat to a body results in mechanical work and have developed in due course of time to provide us respectively several types of modern heat engines-the steam engine, internal combustion engines and gas turbines. We shall now discuss their essential thermodynamics. For any heat engine there are three essential requirements:

(i) **SOURCE:** A hot body, at a fixed temperature T_1 from which the heat engine can draw heat, is called source.

(ii) **SINK:** A cold body, at a fixed lower temperature T_2, to which any amount of heat can be rejected, is called sink.

(iii) **WORKING SUBSTANCE**: The material, which on being supplied with heat, performs mechanical work, is called the working substance.

Thus in a heat engine, the working substance takes in heat from the source, converts a part of it into external work, gives out the rest to the sink and returns to its initial state. This series of operations constitute a cycle.This has been shown in Fig.1. The work can be continuously obtained by performing the same cycle over and over again.

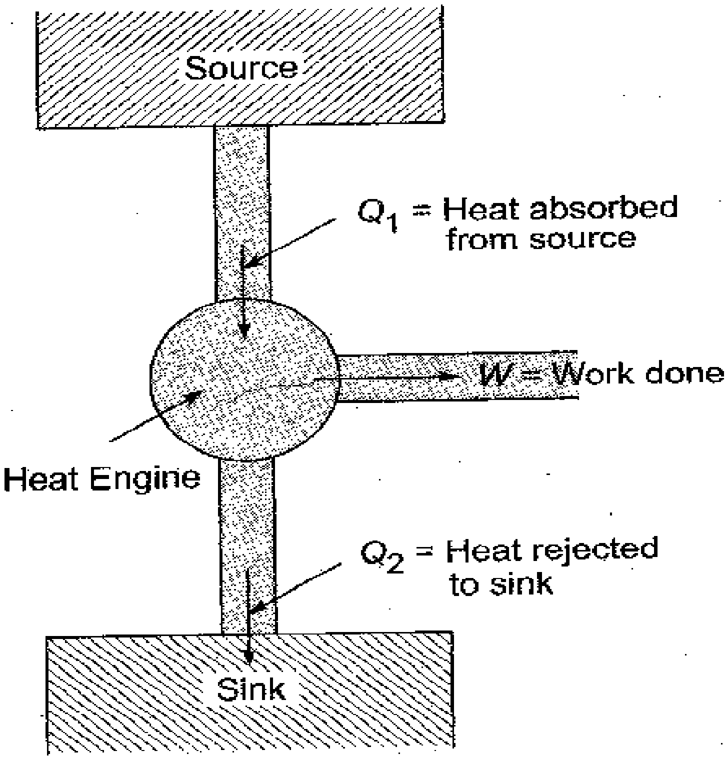

Fig.1: Heat Engine

Let Q be the amount of heat absorbed by the working substance from the source, Q2 that rejected by it to the sink and W the net amount of work done by it. The net amount of heat absorbed by the substance is then Q-Q Remembering that the working substance returns to its initial condition, the change in internal energy du is zero.

By the application of first law of thermodynamics.

$$W = Q_1 - Q_2 \qquad \text{..............(1)}$$

8.3.1 Efficiency of Heat Engine

The thermal efficiency of the engine, η, is defined as the ratio of the net work obtained in the cycle (output) to the heat absorbed by the working substance from the source (input), ie.

$$\eta = \frac{\text{Work output}}{\text{Heat input}}$$

$$= \frac{W}{Q_1}$$

$$= \frac{Q_1 - Q_2}{Q_1}$$

$$\eta = 1 - \frac{Q_2}{Q_1} \qquad \text{..........(2)}$$

From equation(2) it is clear that η will be unity (efficiency 100%) if Q_2 is zero or in other words if an engine could be built to operate in such a way that no heat is at all rejected by the working substance in a cycle, there will be hundred percentage conversion of heat into work.

8.4 REVERSIBLE PROCESS

A reversible process is one which can be retraced in opposite order by slightly changing the external conditions. The working substance in the reverse process passes through all the stages as in the direct process in such a way that all changes occurring in the direct process are exactly repeated in the opposite order and inverse sense and no changes are left in any of the bodies participating in the process or in the surroundings. If heat is absorbed by the substance in the direct process, the same quantity will be given out by it in the reverse process, and if work is done by the substance in the direct process, an equal amount of work will be done on the substance in the reverse process. Thus there is no wastage of energy at all in the reversible process.

Example of Reversible Processes

As an example of a reversible process, consider a gas enclosed in a cylinder, made of perfectly conducting material and immersed in a large tank of water of a constant temperature. Let the gas be compressed very slowly such that its temperature remains unchanged throughout. Obviously to do so small pauses will have to be given in between various small compressions to enable the heat generated by compression to pass out into the surrounding water. If now after reducing considerably the volume of the gas, it is allowed to expand isothermally and the expansions be just as infinitely small as compressions with similar pauses in between, heat will flow in from the enclosing water to compensate for the loss during expansion and will keep the temperature of the gas unchanged. Exactly the same amount of heat will be received during the expansion as will be given up during compession. Thus all the stages of the process are retraced in the opposite direction and inverse order and hence the slow isothermal expansion and compression of a gas is reversible process. In fact all isothermal and adiabatic operations are reversible when carried out very slowly. Similarly an extremely slow contraction or extension of a spring is also reversible if the work done by the spring in each step of infinitesimal contraction is exactly equal to that done on the spring in each corresponding step during extension.

8.5 IRREVERSIBLE PROCESS

The processes which can not be retraced or exactly repeated in the opoosite order by reversing the controlling factors are known as irreversible processes.

Examples of Irreversible Processes

An example of irreversible process is the conduction of heat from a hot body to colder one. Production of heat by friction or by the passage of current through an electrical resistance are also irreversible processes, because heat will again be produced (and not absorbed) if the direction of motion or the direction of flow of current are reversed. The Joule-Thomson effect is also an irreversible process as the fall in temperature take place whether the gas crosses the porous plug in one direction or in the reverse direction. Rapid isothermal and adiabatic changes are irreversible.

8.6 CONDITIONS OF REVERSIBILITY

(i) The substance undergoing a reversible change must not lose heat by conduction, convection or radiation or in overcoming friction. No heat must at all be converted into magnetic or electrical energy. Hence for reversibility complete absence or dissipative effects such as friction, electrical resistance, magnetic hysteresis etc. is a must.

(ii) The changes in the pressure and volume of the working substance must take place at an infinitely slow rate; so that when the substance is receiving heat its temperature differs from the hotter body by only an infinitesimal amount and when it is losing heat the temperature again differs by an infinitesimal amount from the colder body. Thus all reversible processes must take place infinitely slowly.

These conditions are never strictly realized in practice because no mechanical process is frictionless and no insulator or conductor is perfect. Thus rigorous reversibility is an ideal conception while irreversibility is the rule. However, the conditions necessary for reversibility can be fulfilled approximately and such processes may be regarded as reversible within the limits of experimental errors.

8.7 CARNOT’S ENGINE AND CARNOT'S CYCLE

A heat engine is a practical arrangement to convert heat into mechanical work. Sadi Carnot conceived an ideal theoretical engine free from all the imperfectness of actual engines and hence never realized in actual practice. His imaginary engine is, however, taken as a standard against which the performance of actual engines is judged. The plan of Carnot's ideal engine is shown in Fig.2.

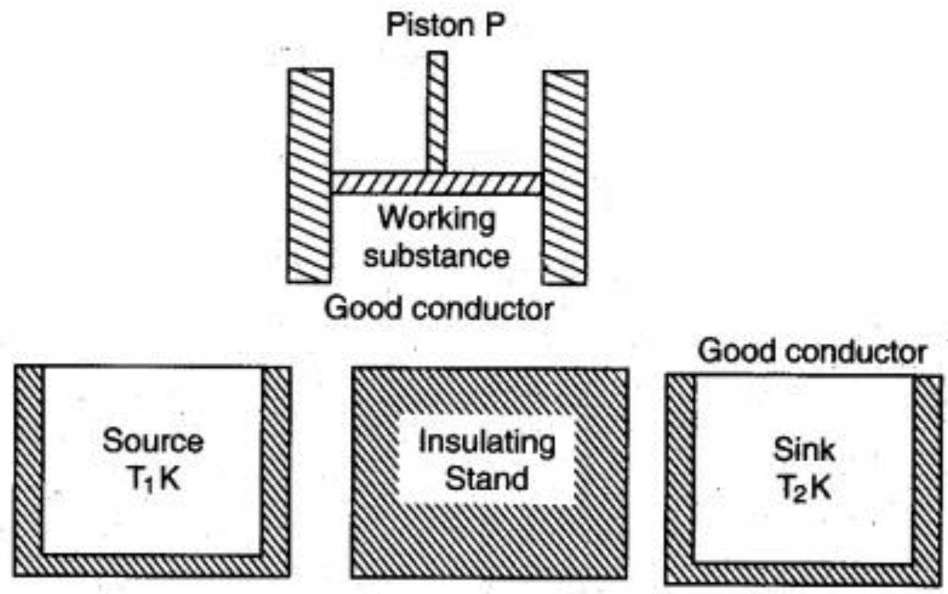

Fig.2: Schematic representation of Carnot Heat Engine

It consists of:

(i) A cylinder with perfectly non-conducting walls but perfectly conducting base containing air (which is supposed to behave like a perfect gas) as the 'working substance' and fitted with a perfectly insulating and frictionless piston upon which weights can be placed.

(ii) A hot body of infinitely large heat capacity maintained at a constant high temperature T_1 absolute serving as the 'source' of heat.

(iii) A cold body of infinitely large heat capacity maintained at a lower constant high temperature T_2 absolute serving as the 'sink'.

(iv) A perfectly insulating platform serving as a 'stand' for the cylinder.

The cylinder may be placed on any of the three bodies (ii), (iii) and (iv) and may be moved from one to the other without friction, i.e., without doing any work.

8.7.1 Carnot's Cycle

The working subtance is subjected to cycle of four operations, consisting of two isothermal operations and two adiabatic operations.

Such a cycle is known as Carnot's cycle and is represented on the P-V (indicator) diagram of Fig.3.

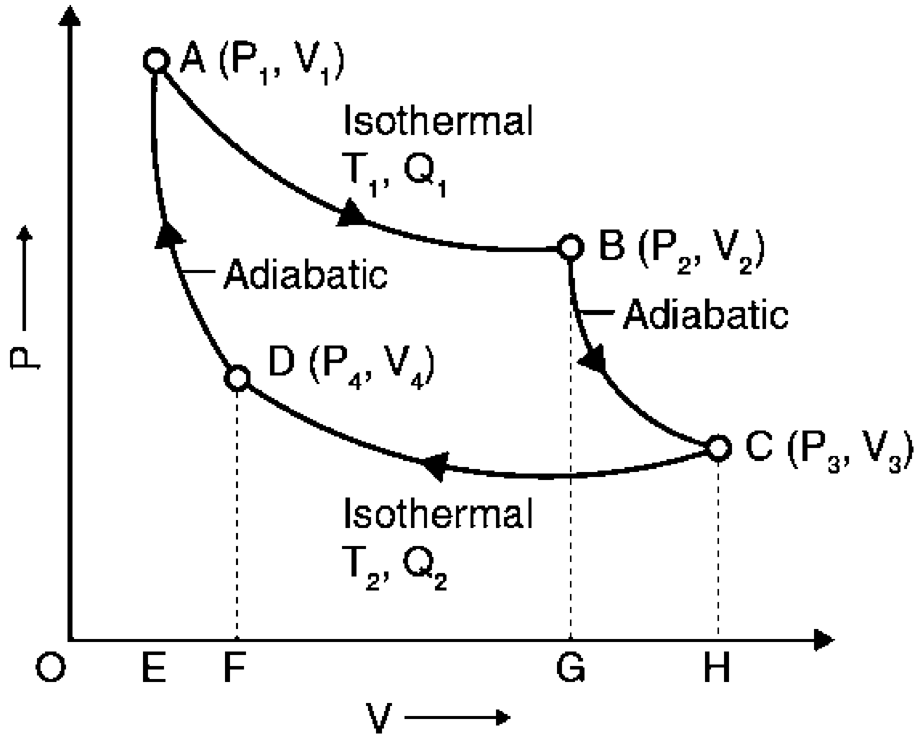

Fig.3: Carnot's Cycle Diagram

Let us now consider the four oprations of the Carnot's cycle. Let the cylinder contain one gm. mol. of the working substance and the original condition of the substance be represented by point A on the indicator diagram, where it has temperature T_1, pressure P_1 and volume V_1.

Operation 1: Isothermal Expansion: The cylinder is placed on the source and the load (pressure) on the piston is slowly decreased. The working substance thus expands doing external work in raising the piston. This would make the substance fall in temperature, but as it is in contact with the source, it takes in necessary heat by conduction through the base to expand isothermally at the constant temperature T_1 of the source. This operation is represented by the isothermal curve AB on the indicator diagram. Let the quantity of heat absorbed in this process be Q_1. Then in accordance with the first law of thermodynamics, Q_1 must be equal to the external work done by the gas in expanding isothermally from A to be B at temperature T_1, (as internal energy remains unchanged) and with the property of indicator diagram is represented by the area ABGEA. Hence

$$Q_1 = W_1 = \int_{V_1}^{V_2} \text{PdV} \quad = RT_1 \int_{V_1}^{V_2} \frac{dV}{V} = RT_1 log_e \frac{V_2}{V_1}$$

$$= \text{Area ABGEA.} \qquad \ldots(3)$$

Operation 2: Adiabatic Expansion: The cylinder is removed from the source, put on the perfectly non-conducting stand and by further decreasing the load on the piston, the substance is allowed to expand. The expansion is completely adiabatic because no heat can enter or leave the substance through the insulating cylinder. The substance performs external work in raising the piston at the expense of its internal energy and its temperature, therefore, falls. The gas is thus allowed to expand adiabatically until its temperature falls to T_2. the temperature of the sink. It has been represented by the adiabatic curve BC on the indicator diagram. If P_3. Va be the pressure and volume of the substance at C, then work done by substance from B to C (adiabatic process)

$$W_2 = \int_{V_2}^{V_3} \mathrm{PdV} \; = K \int_{V_1}^{V_2} \frac{dV}{V^{\gamma}}$$

Already know that for adiabatic process $PV^{\gamma} = K(a\ constant)$

$$= \frac{KV_3^{1-\gamma} - KV_2^{1-\gamma}}{1-\gamma}$$

$$= \frac{\mathrm{P_3V_3 - P_2V_2}}{1-\gamma} \qquad [\because\because \mathrm{P}_2V_2^{\gamma} = \mathrm{P}_3\mathrm{V}_3^{\gamma} = \mathrm{K}]$$

$$= \frac{RT_2 - RT_1}{1-\gamma} \qquad [\because \mathrm{P}_2V_2 = \mathrm{R}T_1 \;, \mathrm{P}_3V_3 = \mathrm{R}T_2\,]$$

$$= \frac{R(T_2 - T_1)}{1-\gamma}$$

= Area BCHGB ...(4)

Operation 3: Isothermal Compression: The cylinder is removed from the non-conducting stand and placed on the sink at temperature T_2. The load on the piston is slowly increased so that the gas is compressed until its pressure and volume become P_4, V_4 represented by the point D. The heat developed due to compression immediately passess into the sink through the conducting base and the temperature of the working substance remains constant at T_2, the temperature of the sink which remains unchanged due to its infinite heat capacity. This compression is represented by the isothermal CD on the indicator diagram. The quantity of heat Q_2 rejected to the sink during this process is equal to the work done W_3 on the working substance in compressing it isothermally from C to D. Hence,

$$Q_2 = W_3 = \int_{V_3}^{V_4} \mathrm{PdV} \; = -RT_2 \int_{V_4}^{V_3} \frac{dV}{V} = -RT_2 log_e \frac{V_3}{V_4}$$

= Area CHFDC. ...(5)

Operation 4: Adiabatic Compression: The cylinder is again transferred to the insulating stand and the load on the piston is again slightly increased so that the substance undergoes a slow adiabatic compression and its temperature rises. This compression is continued until the temperature rises to T_1, and the substance comes back to its original pressure P_1, and volume V_1. Thus the internal energy of the substance is the same as at the beginning of the process. This compression is represented by the adiabatic DA on the indicator diagram. The work done on the substance during this adiabatic compression from D to A is

$$W_4 = \int_{V_4}^{V_1} \mathrm{PdV} \; = K \int_{V_4}^{V_1} \frac{dV}{V^{\gamma}}$$

Already know that for adiabatic process $PV^{\gamma} = K(a\ constant)$

$$= \frac{KV_1^{1-\gamma} - KV_4^{1-\gamma}}{1-\gamma}$$

$$= \frac{P_1V_1 - P_4V_4}{1-\gamma} \qquad [\because P_1V_1^{\gamma} = P_4V_4^{\gamma} = K]$$

$$= \frac{RT_1 - RT_2}{1-\gamma} \qquad [\because P_1V_1 = RT_1 \ , P_4V_4 = RT_2\ ]$$

$$= -\frac{R(T_1-T_2)}{1-\gamma} \qquad \ldots\ldots\ldots..(6)$$

= Area DFEAD.

Work done by the engine per cycle : During the above cycle of operations, the working substance takes in an amount of heat Q1 from the source and rejects Q2 to the sink. Hence the net amount of heat absorbed by the substance

$$= Q_1 - Q_2$$

At the same time, the net work done by the engine in one complete cycle

$$= Area\ ABGEA + Area\ BCHGB - Area\ CHFDC - Area\ DFEAD$$

$$= Area\ ABCHEA - Area\ CHEADC$$

$$= Area\ ABCD.$$

Thus the work done in one cycle is represented on a P-V diagram by the area of the cycle. The net work done by the engine per cycle may also be given as

$$W = W_1 + W_2 + W_3 + W_4$$

$$= RT_1 log_e \frac{V_2}{V_1} + \frac{R(T_2-T_1)}{1-\gamma} - RT_2 log_e \frac{V_3}{V_4} - \frac{R(T_1-T_2)}{1-\gamma}$$

$$W = RT_1 log_e \frac{V_2}{V_1} - RT_2 log_e \frac{V_3}{V_4} \qquad \ldots\ldots\ldots\ldots(7)$$

Now, since points A and D lie on the same adiabatic DA

$$T_1V_1^{\gamma-1} = T_2V_4^{\gamma-1}$$

Then

$$\frac{T_2}{T_1} = \left(\frac{V_1}{V_4}\right)^{\gamma-1} \qquad \ldots\ldots\ldots..(8)$$

The points B and C also lie on the same adiabatic BC.

$$T_1V_2^{\gamma-1} = T_2V_3^{\gamma-1}$$

$$\frac{T_2}{T_1} = \left(\frac{V_2}{V_3}\right)^{\gamma-1} \qquad \ldots\ldots\ldots\ldots\ldots..(9)$$

By equating the equations (8) and (9) , we have

$$\left(\frac{V_1}{V_4}\right)^{\gamma-1} = \left(\frac{V_2}{V_3}\right)^{\gamma-1} \qquad \ldots\ldots\ldots\ldots..(10)$$

$$\frac{V_1}{V_4} = \frac{V_2}{V_3}$$
$$i.e. \frac{V_2}{V_1} = \frac{V_3}{V_4}$$

If we put the above value in equation(7), then we have

$$W = RT_1 log_e \frac{V_2}{V_1} - RT_2 log_e \frac{V_2}{V_1}$$

$$W = R(T_1 - T_2) log_e \frac{V_2}{V_1} \qquad \ldots\ldots\ldots\ldots(11)$$

As we know that the net amount of heat absorbed by the substance in the cycle is $(Q_1 - Q_2)$, and since initial and final states of the substance are the same, its internal energy remains unchanged and hence from first law of thermodynamics

$$W = Q_1 - Q_2$$

$$W = Q_1 - Q_2 = R(T_1 - T_2) log_e \frac{V_2}{V_1} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(12)$$

It is clear from the above equation that heat has been converted into work by the system and any amount of work can be obtained by simply repeating the cycle.

8.7.2 Efficiency of the Engine:

The efficiency of the engine is given by

$$\eta = \frac{\text{Heat converted into work}}{\text{Heat taken in from the source}}$$

$$\eta = \frac{Q_1 - Q_2}{Q_1}$$

$$= \frac{R(T_1 - T_2) log_e \frac{V_2}{V_1}}{RT_1 log_e \frac{V_2}{V_1}}$$

$$\eta = 1 - \frac{Q_2}{Q_1} = \frac{T_1 - T_2}{T_1} = 1 - \frac{T_2}{T_1} \qquad \ldots\ldots\ldots(13)$$

This expression shows that efficiency of the engine depends upon the temperatures T_1 and T_2 of the source and sink respectively and greater the difference between T_1 and T_2, the greater is the efficiency. Since, however, T_1 >(T_1 -T_2), the efficiency is always less than 1 or 100%.

The efficiency may also be expressed in terms of adiabatic expansion ratio p. We have seen above that considering two adiabatics BC and DA, we can get

$$\frac{V_3}{V_2} = \frac{V_4}{V_1}$$

Each one of these ratio is the adiabatic expansion ratio p; and hence

$$\frac{V_3}{V_2} = \frac{V_4}{V_1} = \frac{1}{\rho}$$

Substituting it is equation (8) or (9), we have

$$\frac{T_2}{T_1} = \left(\frac{1}{\rho}\right)^{\gamma-1}$$

And, therefore, efficiency

$$\eta = 1 - \left(\frac{1}{\rho}\right)^{\gamma-1} \qquad \text{...........(14)}$$

8.8 ENTROPY

The concept of entropy (literal meaning 'transformation') was first introduced by Claussius in 1854 while working on the formulation and application of the second law of thermodynamics. It is a very important thermodynamic quantity and has proved very useful in the study of behaviour of heat engines.

Consider a number of isothermals I_1, I_2, I_3... etc. at temperatures T_1, T_2, , T_3, ... etc.. on an indicator diagram (Fig.4). Let A_1, and A_2 be two adiabatics which intersect these isothermals in points A and B, C and D, E and F etc. Then all along the adiabatics A_1 and A_2 there is a change in volume and temperature with change in pressure. Let ABCD and DCEF represents the Carnot's reversible cycle. Considering the cycle ABCD, let Q_1 be the heat absorbed from A to B at temperature T_1 and let Q_2 be the heat rejected from C to D at temperature T_2, then from the theory of a Carnot engine

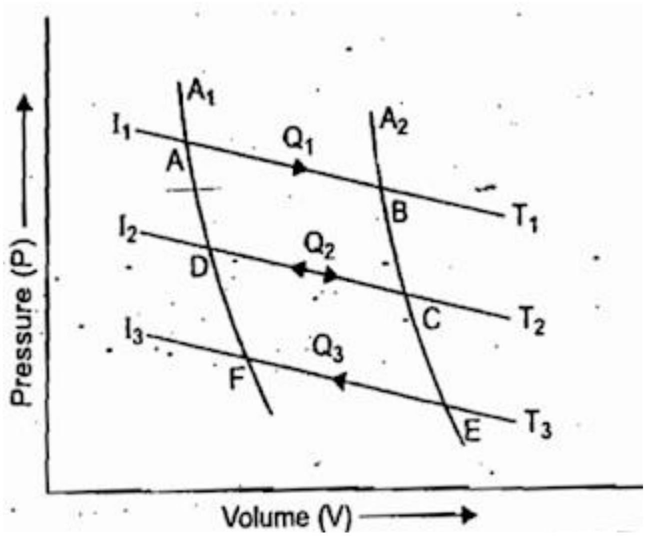

Fig.4: Entropy diagram

Similarly considering the cycle DCEF, if Q_2 be heat drawn at T_2 and Q_3 heat liberated at T_3.

In going from one adiabatic to the other, heat energy is either absorbed or liberated. The amount of heat absorbed or liberated is not constant but depends upon the temperature. Higher the temperature, the more is the heat absorbed or liberated and vice versa. In general, if Q is the amount of heat absorbed or rejected at a temperature T in going from one adiabatic to the other, then

$$\frac{Q}{T} = constant$$

This constant ratio is called the change in entropy between the states represented by the two adiabatics.

Let S_1 and S_2 (arbitrary quantities) be respectively the entropy for the adiabatics A_1 and A_2 then

$$S_2 - S_1 = \frac{Q}{T} = constant \qquad ....(15)$$

If the adiabatics lie very close to each other and dQ is the quantity of heat absorbed or rejected at a temperature T in going from one adiabatic to the other, then change in entropy.

$$dS = \frac{dQ}{T} \qquad ....(16)$$

Hence, in general, the change in entropy in passing from one adiabatic to another

$$\int_{S_1}^{S_2} dS = S_2 - S_1 = \int_{A_1}^{A_2} \frac{dQ}{T} \qquad ....(17)$$

The expression $\int_{A_1}^{A_2} \frac{dQ}{T} = \int_{S_1}^{S_2} dS$ is a function of the thermodynamic coordinates of a system and refers to the value of the functions at the final states minus the value at the initial state. This function is represented by the symbol S and is called entropy. Hence entropy of a system is a function of the thermodynamical coordinates defining the state of the system viz., the

pressure, temperature, volume or internal energy, and its change between two states is equal to the integral of the quantity dQ/T between the states along any reversible path joining them. dS is an exact differential as it is differential of an actual function.
Further it can be easily seen that since during an adiabatic change no heat energy is given to or removed from the system $dQ = 0$, so that the change in entropy $dQ/T = 0$. It means there is no change of entropy during an adiabatic process, or the entropy remains constant during an adiabatic reversible process. It is why the adiabatic curves on the P-V diagram are called as isentropics - curves of constant entropy. Therefore the entropy of substance is that physical quantity which remains constant when the substance undergoes a reversible adiabatic process.

8.8.1 Physical Concept of Entropy

It is difficult to form a physical concept of entropy as there is nothing physical to represent it and it can not be felt like temperature or pressure. But since

$$Change\ in\ entropy = \frac{Amout\ of\ heat\ added\ or\ removed}{Absolute\ temperature}$$

We may say that heat energy has the same dimensions as the product of entropy and absolute temperature. Since the gravitational potential energy of a body is proportional to the product of its mass and height above some zero level hence if we may take temperature (measured from absolute zero) equivalent to height we may regard entropy as analogous to mass or inertia. In this way, we may think of entropy as thermal inertia which bears to heat motion a relation similar to that which mass bears to linear motion or moment of inertia bears to rotational motion.

Unit of entropy

It depends on the unit of heat employed and the absolute temperature. It is measured in calories (or ergs or Joules) per degree Kelvin i.e. cal./K or Joule/K.

8.9 CHANGE OF ENTROPY IN A REVERSIBLE PROCESS

Let us consider a complete reversible process - a Carnot's cycle ABCD shown in Fig.5. In the isothermal expansion from A to B, the working substance absorbs an amount of heat Q_1 at a constant temperature T_1 of the source. When heat is absorbed by the system, Q_1 is positive, and hence entropy change is positive because T is positive. Hence gain in entropy of working substance from A to B = $\frac{Q_1}{T_1}$.(Source loses this heat Q_1at temperature T_1; so its entropy decreases by $(\frac{Q_1}{T_1})$.
During the adiabatic expansion from B to C, there is no change in entropy (since heat is neither taken in nor given out).

During the isothermal compression from C to D, the working substance gives out a quantity of heat Q_2 at constant temperature T_2 of sink and so the loss in its entropy from C to D= $\frac{Q_2}{T_2}$. (The sink gains this heat Q_2 at temperature T_2, so its entropy increases by $\frac{Q_2}{T_2}$.). Again during the adiabatic compression from D to A, there is no change in entropy. Thus the net gain in the entropy of working substance in the whole cycle ABCDA

$$= \frac{Q_1}{T_1} - \frac{Q_2}{T_2}$$

But since in a complete reversible Carnot's cycle

$$\frac{Q_1}{T_1} = \frac{Q_2}{T_2}$$

Therefore

$$\frac{Q_1}{T_1} - \frac{Q_2}{T_2} = 0$$

It means that the total change in entropy of the working substance in a complete cycle of reversible process is zero. Similarly the change in entropy of the combined system of source and sink is also zero. Thus in a cycle of reversible process, the entropy of the system remains unchanged or the change in entropy of the system is zero, i.e.

$$\oint dS = \frac{Q_1}{T_1} - \frac{Q_2}{T_2} = \sum \frac{Q}{T} = 0$$

where the integral sign with a circle refers to a complete cycle.

8.10 CHANGE OF ENTROPY IN AN IRREVERSIBLE PROCESS

Suppose the working substance in an engine performs an irreversible cycle of changes, absorbing an amount of heat Q1 at a temperature T_1 from the source and rejecting the quantity of heat Q_2 at a temperature T_2 of the sink. Then the efficiency of this cycle is given by

$$\frac{Q_1 - Q_2}{Q_1} = 1 - \frac{Q_2}{Q_1}$$

According to Carnot's theorem, this efficiency is less than that of a reversible engine working between the same two temperatures T, and T2 for which

$$\eta = 1 - \frac{T_2}{T_1}$$

Thus

$$1 - \frac{Q_2}{Q_1} < 1 - \frac{T_2}{T_1}$$

Or $\frac{Q_2}{Q_1} > \frac{T_2}{T_1}$ or $\frac{Q_2}{T_2} > \frac{Q_1}{T_1}$

From above we can write as

$$\frac{Q_2}{T_2} - \frac{Q_1}{T_1} > 0$$

Considering the whole system, the source losses the entropy by an amount $\frac{Q_1}{T_1}$and the sink gains an entropy $\frac{Q_2}{T_2}$. Therefore, the net change in entropy for the whole system is $\frac{Q_2}{T_2} - \frac{Q_1}{T_1}$ which is clearly greater than zero or positive. Thus there is an increase in entropy of the system during an irreversible process.

We may make this point still more clear by taking another concrete example for an irreversible process like conduction or radiation of heat. Let a system consist of two bodies at temperatures T1 and T2 respectively, where T_1> T2. Since heat always flows from a higher to a lower temperature, both by conduction and radiation, let Q be the quantity of heat thus transmitted.

Decrease in entropy of hotter body = $\frac{Q}{T_1}$

Increase in entropy of colder body = $\frac{Q}{T_2}$

Therefore, the net increase in entropy of the system

$$= \frac{Q}{T_2} - \frac{Q}{T_1}$$

which is a positive quantity since $T_1 > T_2$ We may, therefore, generalise the result and say that the entropy of a system increases in all irreversible processes. This is known as the law or principle of increase of entropy.

8.11 PRINCIPLE OF INCREASE OF ENTROPY OR DEGRADATION OF ENERGY

We have seen above that the entropy of a system remains constant in reversible cyclic processes but increases inevitably in all irreversible processes. Since a reversible process represents a limiting ideal case, all actual processes are inherently irreversible. It means that as cycle after cycle of operation is performed, the entropy of the system increases and tends to a maximum value. This is the principle of increase of entropy and may be stated as "The entropy of an isolated or self-contained system either increases or remains constant according as the processes it undergoes are irreversible or reversible. Analytically it may be expressed as $dS \geq 0$, where the equality sign refers to reversible processes and the inequality sign to irreversible processes. Therefore, the necessary and sufficient condition of equilibrium of a self contained system is that its entropy should be maximum, for then S can not increase and ds can not be greater than zero.

8.11.1 Entropy and Unavailable Energy

Now since all physical operations in the universe are irreversible, for every such operation performed, a certain quantity of energy of the universe becomes unavailable for useful work and is added to the universe in the form of heat through friction, conduction or radiation. In this way, in a distant future, on account of irreversibility, all energies existing in different froms will be converted into heat energy and will not be available for conversion into mechanical work, i.e., the availale energy of the universe will tend towards zero. It will correspond to a state of maximum entropy and all temperature difference between various bodies of the universe will be equalized due to convection etc. No heat engine will then be able to work in this state, because no heat flow would be possible due to the uniformity of temperature throughout the universe. This is spoken as the principle of degradation of energy and implies that although the total amount of energy is conserved, it is transformed into a form which is unavailable for work. Thus the energy is 'running downhill' and the universe is marching towards a stage of stagnancy to a die a 'heat-death'.

In a reversible process (eg. Carnot cycle)

Q_1 →Heat absorbed at temperature T_1
Q_2→ Heat rejected at temperature T_2

Available energy = Q_1 - Q_2

Unavailable energy = Q2

But in Carnot cycle

$$\frac{Q_1}{T_1} = \frac{Q_2}{T_2}$$

Or

$$Q_2 = T_2 \frac{Q_1}{T_1}$$

Thus at constant temperature T_2, the unavailable energy Q_2 depends upon the value of which is the increase in entropy or reversible process at temperature T_1. Thus entropy is the measure of unavailable energy in a system.

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8.11.2 Entropy and Disorder

With an increase in entropy, the thermal agitation and hence disorder of the molecules of substance increases, i.e., growth of entropy implies a transition from order to disorder. Thus the principle of increase of entropy is intimately connected with the less ordered state of affairs. According to it, a high entropy system should be in great disorder or chaos. Thus the entropy of a substance in gaseous state is more than in the liquid state because the molecules are more free to move about in great disorder in a gas than in a liquid. Moreover the entropy is more in the liquid state than in the solid state, as the Molecules are more free to move in a liquid state than in a solid. Hence when ice is converted into water and then into steam, the entropy and disorder of molecules increase. On the other hand, when the steam is converted into water and then into ice, the entropy and disorder of molecules continually decrease. Thus when temperature of a system is lowered, the amount of entropy and disorder in it decrease. Entropy of a substance is, therefore, said to be a measure of the degree of disorder prevailing among its molecules, just as temperature is a measure of the degree of hotness of a substance. At the absolute zero of temperature the thermal motion completely disappears so that the disorder and hence entropy tend to zero and the molecules of a substance are in perfect order i.e., well arranged (third law of thermodynamics).

We may now summarize the above arguments and say that the entropy of any isolated system increases and approaches, more or less rapidly, to the inert state of maximum entropy. We may recognize this fundamental law of physics to be an inherent tendency of nature to proceed from a more ordered state to a less ordered one or from a less disordered to a more disordered state, or we may state in other words that the ultimate destiny of universe is not order but chaos.

8.12 FORMULATION OF THE SECOND LAW IN TERMS OF ENTROPY

The first law of thermodynamics implies, according to Clausius, that the energy of the universe remains constant (the law of conservation of energy): the second law was summed up by him by saying that the entropy of the universe tends to a maximum (law of increase of entropy). We may, therefore, attempt to enunciate the general statement of second law in terms of entropy in the following words. Every physical or chemical process in nature takes place in such a way so as to increase the entropy of the system.

In order to formulate the second law mathematically, let S_A and S_B be the entropies of a substance in initial and final states A and B respectively, measured from some arbitrary zero. The entropy change is then given by

$$S_B - S_A = \int_A^B \frac{dQ}{T}$$

If, any how, the two states A and B are infinitesimally close, the above equation may be put as

$$dS = \frac{dQ}{T}$$

Or $dQ = T.dS$ which is the required mathematical formulation of second law of thermodynamics.

8.13 SUMMARY

In this unit, you have studied about how to

- Explain Basic function of heat Engine
- Explain heat engine and its various parts
- Differentiate between reversible and irreversible processes
- Evaluate efficiency of heat engines
- Learn about Carnot cycle and Carnot engine
- Entropy and its physical significance
- Change in entropy in reversible and irreversible process

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8.16 TERMINAL QUESTIONS

1. What do you understand by a heat engine and its efficiency?
2. Describe Carnot's cycle. Calculate the work done in cycle of operation. Deduce the efficiency of a Carnot engine.
3. Describe Carnot's cycle and show that all reversible engines working between the same two temperatures have the same efficiency?

4. . Explain entropy. Give its general concept and physical significance. Prove that the entropy of a system increases in an irreversible process.
5. Give the definition of entropy. Prove that the entropy of a system remains constant in a reversible process.
6. "The entropy of a substance is a unique function of its state," explain
7. Prove that the dimensions of entropy are the same as the ratio of heat and temperature.
8. Show that in a reversible cyclic process, the entropy change is zero.
9. Explain the principle of increase of entropy.
10. Discuss T-S diagram and hence establish the expression for efficiency of an engine.
11. For the following processes in an ideal gas state whether the change in entropy is positive, negative or zero?

(i) Reversible adiabatic expansion

(ii) Reversible isothermal compression

(iii) Reversible isobaric expansion

(iv) Joule's free expansion

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UNIT 9 SECOND LAW OF THERMODYNAMICS

CONTENTS

9.1 INTRODUCTION

The first law of thermodynamics is based on the principle of energy conservation. Though it assists us in identifying changes in various entities such as heat, internal energy, work done, and so on, it fails to provide the practical feasibility of a reaction proceeding spontaneously. There are numerous examples of processes that are absolutely permissible under the first law but are not feasible in practise. For example, water in a bucket, cannot freeze on its own. According to the first rule of thermodynamics, a portion of water may absorb heat energy from the rest of the water and evaporate. Due to the loss of some heat energy, the remaining water will freeze. However, this is not feasible in practise. This leads us to the conclusion that more rules must be established in order to fully comprehend the process's spontaneity.

The original formulation of the second law of thermodynamics is extremely concerned with the theoretical characteristics of heat engines, particularly the Carnot cycle. The second law of thermodynamics asserts that processes proceed in a specified direction, whereas the first law does not, i.e. the direction of a specific spontaneous transition may be determined using the second law of thermodynamics. The fact that a procedure satisfies the first law does not guarantee that it will occur. As a result, another principle is required to determine whether the process will occur or not.

Some thermodynamic system parameters, such as internal energy and entropy, cannot be measured directly. As a result, thermodynamic relations can connect these properties to those that can be observed, such as pressure, temperature, compressibility, and so on. Unmeasurable qualities in thermodynamic relationships can be represented as partial derivatives including both intensive and extensive variables. A thermodynamic relation is a rule that can be established by simple thermodynamic reasoning and applies to the majority of systems. Maxwell's relations are useful because they connect quantities that appear unrelated. They enable us to connect data gathered in diverse ways or to replace a difficult measurement with another.

9.2 OBJECTIVES

By the end of this unit, you will be able to –

- Learn about some thermodynamic terms required for second law of thermodynamics
- Know different statements given for second law of thermodynamics.

- Understand different statements of the second law of thermodynamics and their equivalence
- State and prove Carnot theorem
- Explain heat engine and its various parts
- Explain thermal conductivity
- Learn about Maxwell thermodynamic relation
- Understand Maxwell's relations
- Characteristics of black body radiation
- Understand the black body radiation intensity versus wavelength curves: their shape and temperature dependence
- Understand the assumptions made by Max Planck to describe electromagnetic radiation emitted by a black body.
- Quantization of energy and Planck's hypothesis.
- Derivation of Planck’s law.

9.3 LIMITATIONS OF FIRST LAW OF THERMODYNAMICS

Before discussing about the second law of thermodynamics, the limitations of the first law will be discussed.

The first law of thermodynamics establishes the relationship between heat absorbed and work done by a system in a specific process. This law, however, does not specify the direction of heat transfer. This law states that extracting heat from ice by cooling it to a low temperature and then using it to warm water is not conceivable. While it is known from experience that heat cannot be transferred from a lower temperature to a higher temperature until some work is done. While heat naturally flows from higher to lower temperatures.

The first law states that energy does not change with a specified change of condition in an isolated system. However, it does not provide any information on the specified change, such as whether it will occur spontaneously or not.

According to the first law, the system's total energy is conserved, which means that one form of energy can be completely converted into another form of energy. However, as demonstrated by Joule's experiment, heat energy cannot be entirely turned into work energy, whereas work energy can be totally changed into heat energy. As a result, heat and work energy are not interchangeable forms of energy. As a result, the second law of thermodynamics, is required.

The second law of thermodynamics gives the direction of a spontaneous process. It introduces a new concept of entropy that governs the criterion of spontaneity.

9.4 SECOND LAW OF THERMODYNAMICS

The first law of thermodynamics states that mechanical work and heat are equivalent when one is completely converted into the other (W=Q). Therefore the principle of energy conservation applied to a thermodynamic system.

However, if we propose extracting a specific amount of heat from a body and completely converting it into work, the first law is not violated. However, in practice, this is shown to be impossible. If this were conceivable, we could use heat extracted from the ocean's water to propel ships across it. Thus, the first law simply states that if a process occurs, energy will be conserved. It doesn't say whether the process is feasible or not. Similarly, when a hot body and a cold body come into contact, the first law remains intact, regardless of whether heat transfers from the hot to the cold body or vice versa. We know from experience that heat never flows.

Various statements of the Second Law of Thermodynamics

The second law of thermodynamics can be expressed in a variety of forms, with the most notable formulations being by Rudolf Clausius (1854), Lord Kelvin (1851), Max Planck (1926), and others. Although these statements differ, each one predicts the direction in which a change will occur spontaneously. Now we will go over these statements in further detail.

9.4.1 CLAUSIUS STATEMENT OF THE SECOND LAW

Rudolf Clausius, a German physicist, established the second law of thermodynamics in 1850. He discovered the link between heat transfer and work done. His formulation of second law is also known as Clausius statement. “Clausius” statement made use of the “passage of heat" concept. This statement claims that no device can be built that functions in a cycle and produces no effect other than the transfer of heat from a lower temperature (cold region) body to a higher temperature (hot region) body. In other words, heat cannot flow naturally from cold to hot regions unless some external activity is done on the system, such as in a refrigerator.

Clausius also stated that “the entropy increases towards a maximum and the energy of the universe is constant”.

9.4.2 KELVIN-PLANCK’S STATEMENT

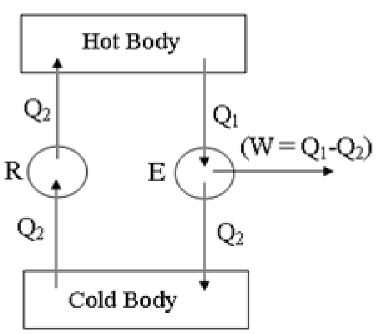
Lord Kelvin gave a definition for the second law of thermodynamics. It states that it is impossible to construct a thermodynamic cycle device that can accept heat from a single heat source and create a net quantity of work. It also claims that no mechanical action can be derived from any portion of matter by cooling it below the temperature of the coldest of the surrounding objects.

If the system violates the Clausius statement, it will also violate the Kelvin statement. As a result, both assertions are identical.

9.4.3 EQUIVALENCE OF KELVIN-PLANCK AND CLAUSSIUS STATEMENTS

We can demonstrate that these two second law statements are equivalent.

Assume there is a refrigerator R (Fig 9.1) that transfers an amount of heat Q2 from a cold body to a hot body without any external energy supply. As a result, it contradicts Clausius' assertion. Assume an engine E operating between the same hot and cold bodies absorbs heat Q1 from the hot body, transforms a portion (W = Q1-Q2) into work, and returns the remaining heat Q2 to the cold body. The presence of engine E alone does not constitute a violation of the law. However, when the refrigerator R and the engine E are joined, they produce a system that absorbs heat Q1-Q2 from the hot body and transforms it entirely into work without giving any amount to the cool body. This clearly contradicts the Kelvin-Planck statement.



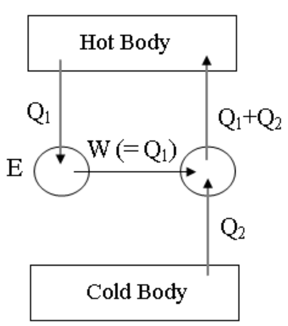


Figure 9.1: Block diagram of a refrigerator R. Figure 9.2: Block diagram of Engine E.

Similarly, suppose there is an engine E (Fig. 9.2) that takes in an amount of heat Q1 from a hot body and entirely converts it into work W(=Q1) without transferring any heat to the cold body. It contradicts the Kelvin-Planck statement. Assume a refrigerator R operating between the same hot and cold bodies absorbs heat Q2 from the cold body, has work W (=Q1) done on it by an external source, and returns heat Q1+Q2 to the hot body. The refrigerator R alone does not constitute a violation of the law. However, when E and R are combined, they create a mechanism that transmits a certain quantity of heat Q2 from a cold body to a hot body with no external energy source. This definitely contradicts Clausius' assertion.

The second law of thermodynamics is an addition to the first. The first law simply states that no device may release more energy than it receives. It makes no mention of any limitations or conditions required for energy supply. The second law, on the other hand, does it. For example, heat absorbed by a substance cannot be completely converted into work, nor can heat move spontaneously from a colder to a hotter body. These occurrences are not governed by the first law, but they are disallowed by the second.

Application of the Second Law of Thermodynamics

Here are several applications and uses for the Second Law of Thermodynamics:

According to the law, heat always flows from a warmer body to a colder body. This rule applies to all heat engine cycles, including Otto, Diesel, and others, as well as all working fluids used in the engines. As a result of this rule, automobiles have developed.

Refrigerators and heat pumps that employ the Reversed Carnot Cycle are further applications of this concept. If you want to transfer heat from a lower temperature body to a higher temperature body, you have to provide external work. The original Carnot Cycle, in contrast to the Reversed Carnot Cycle, which utilises effort to transfer heat from a lower-temperature reservoir to a higher-temperature reservoir, uses heat to perform work.

Limitations of Second Law of Thermodynamics

Let us now discuss the flaws or shortcomings of the Second Law of Thermodynamics:

The second law of thermodynamics is a concept that restricts the occurrence of many processes that we know from experience do not occur while being authorised by other physical laws. For example, water in a glass at room temperature never spontaneously cools to form ice cubes, releasing energy into the environment.

The second rule predicts the end of the universe by implying that the cosmos will end in a condition of "heat death," in which everything is the same temperature. When everything at the same temperature, no work can be done and all energy is lost as random atom and molecule motion, which is the most severe level of disorder.

9.5 CARNOT THEOREM

The second law of thermodynamics provides two key implications that may be combined into a theorem known as Carnot's theorem. "The efficiency of a Carnot reversible engine is maximum and is independent of the nature of the working substance," according to this theorem.

Or

"All reversible heat engines operating between the same two temperatures have the same efficiency, and no irreversible heat engine operating between the same two temperatures can be more efficient than Carnot's reversible heat engine."

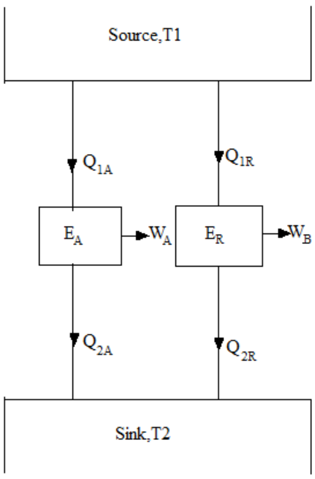


Figure 9.3: Block Diagram of two heat engines, E_A and E_R, between a source at temperature T_1 and a sink at temperature T_2.

Now we will prove this Carnot Theorem. Consider the operation of two heat engines, E_A and E_R, between a source at temperature T_1 and a sink at temperature T_2 (Fig. 9.3). Let E_A represent any heat engine and E_R represent a reversible heat engine. Assume EA's efficiency A is larger than E_R's efficiency ηR. To show the Carnot theorem, we must contradict our assumption. Let the rates of working of the engine E_A be Q_{1A} and that of E_R be Q1R such that-

$Q_{1A}=Q_{1R}=Q_1$

As assumed, $\eta_A > \eta_R$

As we all know,

$$\eta = \frac{Wnet}{Q1}$$

So we can write, $\frac{W_A}{Q_{1A}} > \frac{W_R}{Q_{1R}}$

Therefore, $W_A > W_R$ $(\because QA = Q_R)A$

Let us reverse E_R. Since E_R is a reversible heat engine, therefore, the magnitude of heat transferred and work done will remain the same but their directions will reverse (Fig. 9.4).

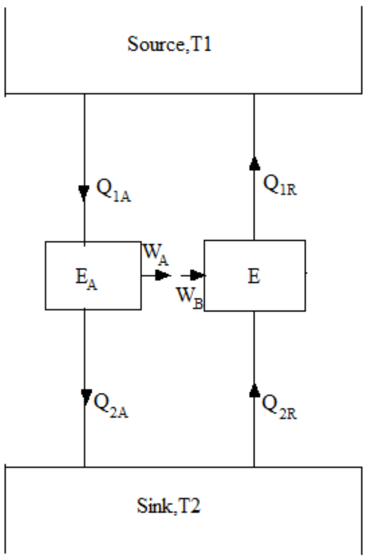


Figure 9.4

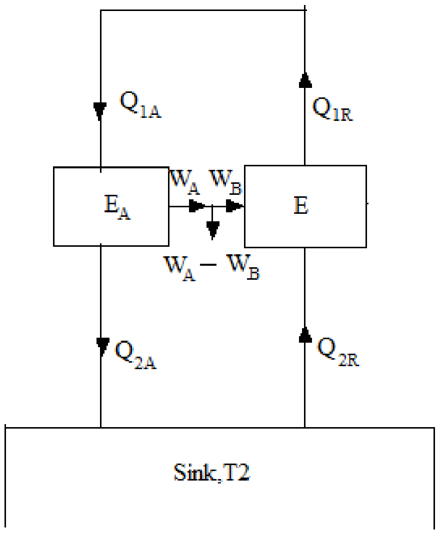


Figure 9.5

Since $W_A > W_R$ some part of W_A which is equal to W_R in magnitude can be fed to drive the reversed heat engine E_R. Since, $Q_{1A}=Q_{1R}=Q_1$, the heat discharged by E_R may be supplied to E_A thus the source may be eliminated. The net result is that E_A and E_R together constitute a heat engine which operating in a cycle produces a net work done W_A-W_R (Fig. 9.5) while exchanging heat with a single reservoir at temperature T_2, thus violating the Kelvin-Planck statement. Hence our assumption is wrong.

Therefore, $\eta_A < \eta_R$ and this proves the Carnot theorem.

9.6 THERMAL CONDUCTIVITY

Thermal conductivity is a transfer of heat from one portion of the body to another as a result of a temperature gradient To determine thermal conductivity, draw three parallel lines at E, X, and F that are normal to the direction of heat flow and separated by mean free path; if two temperatures are equal, $T_1=T_2$, there is no exchange of energy (Figure 9.6).

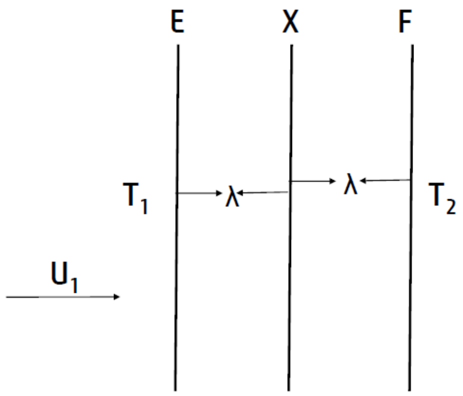


Figure 9.6: Heat conduction from one part of body to another.

If T_1 is greater than T_2, there is an exchange of energy from E to F, therefore the number of electrons per unit area per unit time is $\frac{nu}{6}$ and each electron has energy $\frac{mu_1^2}{2}$.

Thus, Energy transferred from E to F

$$=\frac{nu}{6}\frac{mu_1^2}{2}$$

$$=\frac{nu}{6}\frac{3k_BT_1}{2}$$

$$=\frac{1}{4}nuk_BT_1$$

Likewise, the energy transferred from F to E

$$=\frac{1}{4}nuk_BT_1$$

As a result, the net energy transferred from E to F per unit area per unit time is calculated.

$$=\frac{1}{4}nuk_B(T_1-T_2)$$

As a result, the net energy transferred from E to F per unit area per unit time is calculated.

$$=\frac{K(T_1-T_2)}{2\lambda}$$

On solving

$$K\frac{(T_1-T_2)}{2\lambda}=\frac{1}{4}nuk(T_1-T_2)$$

$$K=\frac{1}{2}\lambda nuk_B \qquad (1)$$

where k_B is Boltzmann constant.

Dividing (1) by $\sigma=\frac{ne^2\lambda u}{6kT}$, we get

$$\frac{K}{\sigma}=\frac{1}{2}\lambda nuk_B\,/\,\frac{ne^2\lambda u}{6kT}$$

or $\frac{K}{\sigma}=3(k_B/e)^2T$ (2)

or $\frac{K}{\sigma}\propto T$, or $\frac{K}{\sigma}=LT$.

This is known as the Wiedemann-Franz relation, and the proportionality constant L is known as the Lorenz number. Thermal conductivity is relatively high in metals, and metals that are excellent electrical conductors are also excellent thermal conductors. Metals' thermal and electrical conductivities are proportionate at a given temperature, however increasing the temperature improves thermal conductivity while decreases electrical conductivity. The Wiedemann-Franz Law considers this conduct.

The relationship is qualitatively based on the fact that both heat and electrical transfer involve free electrons in the metal. Thermal conductivity improves with average particle velocity because it increases energy forward transport. However, as particle velocity rises, electrical conductivity reduces because collisions deflect electrons from forward charge transfer. This indicates that the thermal to electrical conductivity ratio is related to the kinetic temperature and is determined by the average velocity squared.

9.7 DERIVATION OF THE HEAT EQUATION IN ONE DIMENSION

Before deriving heat equation one should have the brief idea about heat. So, at first we will discuss about the heat and then derive the heat equation in one dimension.

Heat

Heat is a type of energy that passes from one medium to another, and it usually travels from the conductor's hotter part to its cooler portion. There are several methods for transferring heat depending on the medium of the conductor. Heat is transported through solids by conduction, liquids and gases via convection, and electromagnetic waves via heat radiation. The one-dimensional heat equation is a partial differential equation that represents how the distribution of heat changes over time in a solid material as it spontaneously flows from higher temperature to lower temperature, which is a specific instance of diffusion.

9.7.1 DERIVATION OF THE HEAT EQUATION IN ONE DIMENSION

A rod of unlimited length can be used to demonstrate the derivation of the heat equation in one dimension. The heat equation for the given rod will be a parabolic partial differential equation, which describes the distribution of heat in a rod over the period of time. Heat energy is transmitted from the conductor's hooter region to the conductor's lower region.

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

where, α is a real coefficient of the equation which represents the diffusivity of the given medium.

Derivation of the Heat Equation in One Dimension

The quantity of heat energy necessary to increase the temperature of the supplied rod by ∂T degrees is shown by CM . ∂T.

This is known as the conductor's specific heat. Where, C is the conductor's positive physical constant of heat and M is the mass of the conductor.

The rate at which heat energy is transferred in the conductor's surface is exactly proportional to the surface area and temperature gradient at the conductor's surface, and this constant of proportionality is known as thermal conductivity of heat, indicated by K.

Consider a finite-length rod with a cross-sectional area A and a mass density ρ.

The function's temperature gradient is expressed as:

$$\frac{\partial T}{\partial x}(x+dx,t)$$

The rate of heat energy transfer from the provided rod's right end is stated as

$$\text{KA}\frac{\partial T}{\partial x}(x+dx,t)$$

The rate of heat energy transfer from the left end is denoted as

$$\text{KA}\frac{\partial T}{\partial x}(x,\text{ t})$$

Because the temperature gradients are positive from both ends, the conductor's temperature must rise.

As the heat flows from the hot region to a cold region of the given rod, heat energy should enter from the right end of the rod and transferred to the left end of the rod.

So, according to the requirement, the equation is as follows:

$$KA\frac{\partial T}{\partial x}(x+dx,t)-KA\frac{\partial T}{\partial x}(x,t)dt$$

The temperature change in the provided rod may now be expressed as

$$\frac{\partial T}{\partial x}(x,t)dt$$

The rod of the mass can be given as:

Density = mass/volume

$$\rho = \frac{M}{A}.dx$$

$$M = \rho A.dx$$

The heat equation may be written as

$$C\rho A dx \frac{\partial T}{\partial x}(x,t)dt = KA\frac{\partial T}{\partial x}(x+dx,,t) - \frac{\partial T}{\partial x}(x,t)dt$$

Dividing both sides of the above equation by dx and dt and taking limits of it dx and it ->0,

$$C\rho A\frac{\partial T}{\partial x}(x,t) = KA\frac{\partial^2 T}{\partial x^2}(x,t)$$

The equation will be,

$$\frac{\partial T}{\partial x}(x,t) = \alpha^2\frac{\partial^2 T}{\partial x^2}(x,t)$$

Where,

$\alpha^2 = K / C\rho$ is the thermal diffusivity of the given rod.

Hence the above-derived equation is the Heat equation in one dimension.

9.7.2 Application of the Heat Equation

The heat equation is used to change automobile engines since it tells you about the specific heat of the conductor, which provides you an idea about the engine's rate of heat absorption and ability to hold the heat.

The hot water bag is most commonly used in the medical industry to provide pain treatment to patients. Heat is transported from the hotter to the cooler area in this situation.

9.8 MAXWELL'S THERMODYNAMIC RELATIONS

If we know the mass of a homogeneous system and any two of the thermodynamic variables P, V, T, U, and S, we may calculate its state entirely. Thus, if V and T are provided, the internal energy U of a system is totally defined. U, in other words, is a function of the two variables V and T. There are specific relationships among the five thermodynamic variables, four of which are important and are known as "Maxwell's thermodynamic relations." Now let us deduce these relationships.

Now from the first law of thermodynamics

$$dQ = dU + dW$$

$$dQ = dU + PdV \qquad \text{(Since } dW = PdV\text{)}$$

or $$dU = dQ - PdV$$

and from the second law of thermodynamics

$$dQ = TdS$$

Substituting the value of dQ in first equation

$$dU = TdS - PdV \tag{3}$$

Let U, S and V be the functions of two independent variables x and y. [Here x and y may be any two variables out of S, T, P and V], then

$$dU = \left(\frac{\partial U}{\partial x}\right)_y dx + \left(\frac{\partial U}{\partial y}\right)_x dy$$

$$dS = \left(\frac{\partial S}{\partial x}\right)_y dx + \left(\frac{\partial S}{\partial y}\right)_x dy$$

$$dV = \left(\frac{\partial V}{\partial x}\right)_y dx + \left(\frac{\partial V}{\partial y}\right)_x dy$$

Substituting these values of dU, dS and dV in equation (3), we get

$$\left(\frac{\partial U}{\partial x}\right)_y dx + \left(\frac{\partial U}{\partial y}\right)_x dy = T\left[\left(\frac{\partial S}{\partial x}\right)_y dx + \left(\frac{\partial S}{\partial y}\right)_x dy\right] - P\left[\left(\frac{\partial V}{\partial x}\right)_y dx + \left(\frac{\partial V}{\partial y}\right)_x dy\right]$$

or $$\left(\frac{\partial U}{\partial x}\right)_y dx + \left(\frac{\partial U}{\partial y}\right)_x dy = \left[T\left(\frac{\partial S}{\partial x}\right)_y - P\left(\frac{\partial V}{\partial x}\right)_y\right]dx + \left[T\left(\frac{\partial S}{\partial y}\right)_x - P\left(\frac{\partial V}{\partial y}\right)_x\right]dy$$

Equating the coefficients of dx and dy on both sides, we have

$$\left(\frac{\partial U}{\partial x}\right)_y = T\left(\frac{\partial S}{\partial x}\right)_y - P\left(\frac{\partial V}{\partial x}\right)_y \tag{4}$$

$$\left(\frac{\partial U}{\partial y}\right)_x = T\left(\frac{\partial S}{\partial y}\right)_x - P\left(\frac{\partial V}{\partial y}\right)_x \tag{5}$$

Differentiating equation (4) with respect to y and equation (5) with respect to x, we get

$$\frac{\partial^2 U}{\partial y \cdot \partial x} = \left(\frac{\partial T}{\partial y}\right)_x \left(\frac{\partial S}{\partial x}\right)_y + T\frac{\partial^2 S}{\partial y \cdot \partial x} - \left(\frac{\partial P}{\partial y}\right)_x \left(\frac{\partial V}{\partial x}\right)_y - P\frac{\partial^2 V}{\partial y \cdot \partial x}$$

$$\frac{\partial^2 U}{\partial x \cdot \partial y} = \left(\frac{\partial T}{\partial x}\right)_y \left(\frac{\partial S}{\partial y}\right)_x + T\frac{\partial^2 S}{\partial x \cdot \partial y} - \left(\frac{\partial P}{\partial x}\right)_y \left(\frac{\partial V}{\partial y}\right)_x - P\frac{\partial^2 V}{\partial x \cdot \partial y}$$

As dU is a perfect differential, therefore,

$$\frac{\partial^2 U}{\partial y \cdot \partial x} = \frac{\partial^2 U}{\partial x \cdot \partial y}$$

$$\left(\frac{\partial T}{\partial y}\right)_x \left(\frac{\partial S}{\partial x}\right)_y + T\frac{\partial^2 S}{\partial y \cdot \partial x} - \left(\frac{\partial P}{\partial y}\right)_x \left(\frac{\partial V}{\partial x}\right)_y - P\frac{\partial^2 V}{\partial y \cdot \partial x}$$

$$= \left(\frac{\partial T}{\partial x}\right)_y \left(\frac{\partial S}{\partial y}\right)_x + T\frac{\partial^2 S}{\partial x \cdot \partial y} - \left(\frac{\partial P}{\partial x}\right)_y \left(\frac{\partial V}{\partial y}\right)_x - P\frac{\partial^2 V}{\partial x \cdot \partial y} \qquad (6)$$

Since dS and dV are also perfect differentials, we have

$$\frac{\partial^2 S}{\partial y \cdot \partial x} = \frac{\partial^2 S}{\partial x \cdot \partial y} \quad \text{and} \quad \frac{\partial^2 V}{\partial y \cdot \partial x} = \frac{\partial^2 V}{\partial x \cdot \partial y}$$

Therefore, equation (6) becomes:

$$\left(\frac{\partial T}{\partial y}\right)_x \left(\frac{\partial S}{\partial x}\right)_y - \left(\frac{\partial P}{\partial y}\right)_x \left(\frac{\partial V}{\partial x}\right)_y = \left(\frac{\partial T}{\partial x}\right)_y \left(\frac{\partial S}{\partial y}\right)_x - \left(\frac{\partial P}{\partial x}\right)_y \left(\frac{\partial V}{\partial y}\right)_x \qquad (7)$$

This is the general expression for Maxwell's thermodynamic relations. In place of the independent variables x and y, any two of the four variables S, T, P and V can be substituted so that there may be one mechanical variable (P or V) and one thermal variable (S or T). Thus there may be four sets of possible substitutions (S, V), (T, V), (S, P) and (T, P), providing the four Maxwell's thermodynamic relations.

Maxwell's First Relation: Substitute x = S and y = V in equation (7), so that

$$\frac{\partial S}{\partial x} = 1, \ \frac{\partial S}{\partial y} = 0$$

$$\frac{\partial V}{\partial x} = 0, \ \frac{\partial V}{\partial y} = 1$$

Putting these values in equation (7), we get

$$\left(\frac{\partial T}{\partial y}\right)_x = -\left(\frac{\partial P}{\partial x}\right)_y$$

But $\partial y = \partial V$ (as y = V) and $\partial x = \partial S$ (as x =S). Hence

$$\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial P}{\partial S}\right)_V \qquad \text{(i)}$$

This is Maxwell's first thermodynamic relation.

Maxwell's Second Relation: Substitute x = T and y = V in equation (7), so that

$$\frac{\partial T}{\partial x} = 1, \ \frac{\partial T}{\partial y} = 0$$

$$\frac{\partial V}{\partial x} = 0, \ \frac{\partial V}{\partial y} = 1$$

Putting these values in equation (7), we get

$$0 = \left(\frac{\partial S}{\partial y}\right)_x - \left(\frac{\partial P}{\partial x}\right)_y$$

or $$\left(\frac{\partial S}{\partial y}\right)_x = \left(\frac{\partial P}{\partial x}\right)_y$$

But $\partial y = \partial V$ (as y = V) and $\partial x = \partial T$ (as x =T). Hence

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V \qquad \text{(ii)}$$

This is Maxwell's second thermodynamic relation.

Maxwell's Third Relation: Substitute x = S and y = P in equation (7), so that

$$\frac{\partial S}{\partial x} = 1, \ \frac{\partial S}{\partial y} = 0$$

$$\frac{\partial P}{\partial x} = 0, \ \frac{\partial P}{\partial y} = 1$$

Putting these values in equation (7), we get

$$\left(\frac{\partial T}{\partial y}\right)_x - \left(\frac{\partial V}{\partial x}\right)_y = 0$$

$$\left(\frac{\partial T}{\partial y}\right)_x = \left(\frac{\partial V}{\partial x}\right)_y$$

But $\partial y = \partial P$ (as y = P) and $\partial x = \partial S$ (as x = S). Hence

$$\left(\frac{\partial T}{\partial P}\right)_S = \left(\frac{\partial V}{\partial S}\right)_P \qquad \text{(iii)}$$

This is Maxwell's third thermodynamic relation.

Maxwell's Fourth Relation: Substitute x = T and y = P in equation (7), so that

$$\frac{\partial T}{\partial x} = 1, \; \frac{\partial T}{\partial y} = 0$$

$$\frac{\partial P}{\partial x} = 0, \; \frac{\partial P}{\partial y} = 1$$

Putting these values in equation (7), we get

$$-\left(\frac{\partial V}{\partial x}\right)_y = \left(\frac{\partial S}{\partial y}\right)_x$$

But $\partial y = \partial P$ (as y = P) and $\partial x = \partial T$ (as x = T). Hence

$$\left(\frac{\partial S}{\partial P}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_P \qquad \text{(iv)}$$

This is Maxwell's fourth thermodynamic relation.

The four main thermodynamic relations are (i), (ii), (iii), and (iv), and any of these relations, depending on its applicability, can be employed to solve a given problem.

9.9 CLAUSIUS-CLAPEYRON LATENT HEAT EQUATION

The second thermodynamic relation of Maxwell is represented as:

$$\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V$$

By Multiplying by T both sides, we have

$$T\left(\frac{\partial S}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V$$

But, $T\partial s = \partial Q$ (from second law of thermodynamics). Hence

$$\left(\frac{\partial Q}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V$$

At constant temperature, $\left(\frac{\partial Q}{\partial V}\right)_T$ denotes the amount of heat absorbed or released per unit change in volume. This means that under constant temperature, heat received or released causes just a change in the volume of the material. As a result, the amount of heat absorbed or expelled at constant temperature must be the latent heat, and the volume change must be attributable to a change in state. Let L be the latent heat when a unit mass of material changes its volume from V_1 to V_2 at constant temperature, then

$$\partial Q = L \text{ and } \partial V = V_2 - V_1$$

Substituting these values in the above expression

$$\left(\frac{L}{V_2 - V_1}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V$$

or $$\frac{L}{V_2 - V_1} = T\frac{dP}{dT}$$

or $$\frac{dP}{dT} = \frac{L}{T\,(V_2 - V_1)}$$

This is the Clausius-Clapeyron latent heat equation.

9.10 BLACK BODY AND BLACK BODY RADIATION

A totally black-body is one that absorbs all incident heat radiations of any wavelength. It does not reflect or transmit any of the incident radiations and hence looks black whatever the colour (wavelength) of the incident radiation.

When a black-body is put in an isothermal enclosure, it will emit the whole radiation of the enclosure after it has reached thermal equilibrium with it. These radiations are unaffected by the composition of the material. Clearly, the radiation from an isothermal container is similar to that from a black-body at the same temperature. As a result, the heat radiations in an isothermal enclosure are referred to as black-body radiation. In practise, no material has all of the qualities of a black-body. Lamp-black and platinum black are quite similar to the colour black. However, bodies that are near to a completely black-body can be built. Ferry's and Wien's black-bodies are two such instances. We'll go over everything in more depth later.

9.11 ENERGY DISTRIBUTION IN BLACK BODY RADIATION

For many years, black body radiation was unidentified. This section explores how the problem was solved, leading to the discovery of new physical rules that serve as the foundation of quantum mechanics.

Lummer and Pringsheim made the first attempt in 1899. As shown in Figure 9.7, they plotted some curves between E (spectral emissive power) and (wavelength) at various temperatures. These charts are known as Black Body radiation spectral energy distribution curves. The figures show that the energy of black body radiation is not distributed uniformly throughout all wavelengths of light. The graph indicates that certain wavelengths get more energy than others.

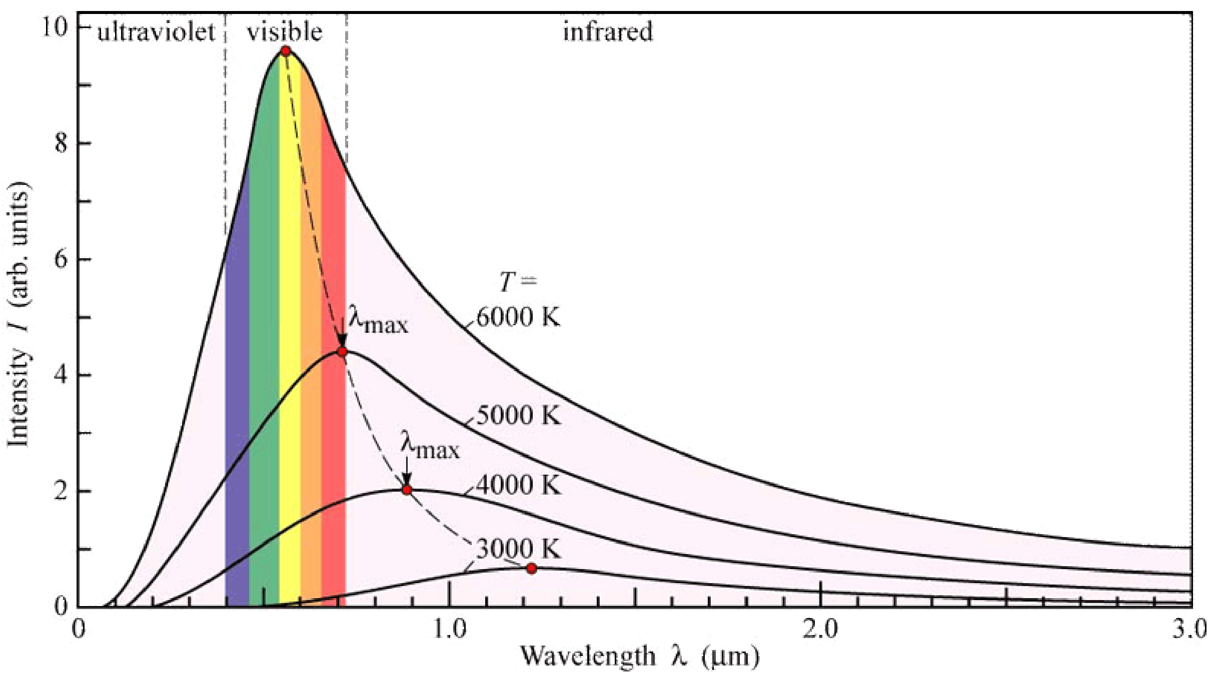


Figure 9.7: spectral energy distribution of black-body radiation versus wavelength

9.11.1 EXPERIMENTAL OBSERVATIONS AND CHARACTERISTICS OF BLACK-BODY RADIATION

1. It is obvious from the figure that the graph is continuous, which implies that radiation for all wavelengths is emitted at all temperatures, although the spectrum emissive strength varies with wavelength. In other words, the distribution of energy in a black-body radiation spectrum is not uniform.

2. The spectral energy density E_λ for each λ increases with temperature, or when an object's temperature rises, it emits more energy at all wavelengths.

3. For a certain temperature, E_λ initially increases with λ, but after reaching a predetermined maximum value, it decreases. The greatest value is represented by $E_{\lambda m}$ and the wavelength at which E_λ is maximum is marked by λ_m.

4. As seen from the graph, the wavelength (λ_m) corresponding to maximum emission, shifts towards lower wavelength with increase in temperature. It was Wein who first discovered mathematically that

$$\lambda_m \propto \frac{1}{T}$$

or $$\lambda_m = \frac{b}{T}$$

or $$\lambda_m T = b \quad \text{(constant)}$$

Where b is the Wein's constant, which has a value of 2.968 X 10^{-3} metre kelvin.

The above equation is known as wein's displacement law. This is an essential rule because it allows us to calculate the temperature of distant hot entities such as stars.

Wein's displacement law may also be represented in terms of frequency as:

$$\nu_m = \frac{cT}{b}$$

The graph also shows that the value corresponding to the peak of the curve grows significantly with temperature. It was found that

$$E\lambda_m \propto T^5$$

6. The total energy emitted by the body at a given temperature is represented by the area under the curve, and the area under the curve at a given temperature is given mathematically by

$$\int_0^\infty E_\lambda d\lambda$$

This is the total emissive power of a Black Body. It was found that area under the curve is directly proportional to the fourth power of absolute temperature, hence

$$E \propto T^4$$

or, $$E = \sigma T^4$$

Where σ is Stephan's constant and has the value

$$\sigma = 5.67 \times 10^{-8} \text{ watt/m}^2/\text{K}^4$$

This law is known as Stephan-Boltzmann's law.

The black-body spectrum is always smaller on the left (i.e., on the shorter wavelength or higher frequency side).

The black-body spectrum is determined only by the body's temperature and not by its composition. If their temperatures are the same, an iron bar, a ceramic pot, and a piece of charcoal will all radiate the same black-body spectrum.

9.11.2 BLACK-BODY

When a smooth surface completely reflects all the incident rays, as is approximately the case with many metallic surfaces, it is termed 'reflecting'. When a rough surface reflects all incident rays completely and uniformly in all directions, it is called 'white'. A rough surface having the property of completely absorbing the incident radiation is described as 'black'.

A perfectly black-body is an idealized physical body which absorbs all the radiations that fall on it, irrespective of the wavelength or angle of incidence.

A black body in thermal equilibrium has two notable properties: It is an ideal emitter: at every frequency, it emits as much energy as or more energy than any other body at the same temperature. It is a diffuse emitter, the energy is radiated isotropically, independent of direction.

The blackbody radiation spectrum shows three significant properties of blackbodies:

A black-body with a temperature greater than absolute zero produces energy in all wavelengths stretching to infinity (curves never intersect on the x-axis). A hotter black-body emits more energy across all wavelengths than a cooler one. The shorter the wavelength at which the highest energy is radiated, the greater the temperature.

9.11.3 QUANTUM THEORY OF RADIATION

In 1901, Max Planck postulated the quantum theory of radiation. According to his theory, radiation energy is always in the form of small bundles of energy called quanta, indicating that energy is absorbed or released discontinuously. Each quantum has a fixed energy that is determined by the frequency of the radiation, as provided by the relation

$E=h\nu$

Here, E represents the energy of each quantum in joules, f represents the frequency of radiation in s^{-1}, and h represents the Planck constant, h=6.62610-34J-s.

also $E=hc\omega$

where ω is known as wave number ($\omega=1/\lambda\ m^{-1}$)

A body's energy released or absorbed is always a whole multiple of a quantum, implying that a body cannot absorb or emit energy in fractions of a quantum. This is referred to as energy quantization.

9.12 PLANCK RADIATION FORMULA

Planck proposed the following hypothesis in order to develop a theory/law that may properly explain the distribution of energy in a black body radiation:

1. A blackbody radiation chamber is composed of a number of oscillating particles (of molecular dimensions) known as Planck's oscillators or Planck's resonators, which are made up of harmonic oscillators or resonators (energy emitters).

An oscillator emits radiation of frequency ν when it drops from one energy state to the next lower one, and it jumps to the next higher state when it absorbs radiation of frequency ν. Each discrete bundle has energy hν or multiples of hν. It is given by

$\varepsilon_n = nh\nu$

where, n = 0,1,2,3, ….

and h (Planck's constant) = 6.6 x 10^{-34} J-s

As a result, radiation energy is discrete rather than continuous. Oscillator energy levels are so like 0, 0, hν, 2hν, 3hν, 4hν………, nhν.

9.12.1 DERIVATION OF PLANCK'S RADIATION LAW

Assume that the body contains N_0, N_1, N_2, N_3, ……..N_n vibrating particles (Planck's resonator). The energy of the aforesaid particles may be expressed as 0, ε, 2ε, 3ε, 4ε,……..nε according to Planck's hypothesis.

As a result, the total number of vibrating particles is

$N=N_0+N_1+N_2+N_3+\ldots..N_n$

Similarly total energy of the body

$E=0+\varepsilon+2\varepsilon+3\varepsilon+4\varepsilon+\ldots\ldots..n\varepsilon$

Therefore, average energy of a particle is given by

$$\bar{\varepsilon} = \frac{E}{N}$$

The number of particles in the n^{th} oscillating system may be represented as according to Maxwell's distribution law.

$$N_n = N_0 e^{-\frac{n\varepsilon}{kT}}$$

In the above equation ε represents the average energy per oscillator, k is the Boltzmann constant and T is the absolute temperature.

Extending Maxwell distribution formula to the present system, the total number of particles can be written as

$$N = N_0 + N_0 e^{-\frac{\varepsilon}{kT}} + N_0 e^{-\frac{2\varepsilon}{kT}} + N_0 e^{-\frac{3\varepsilon}{kT}} + \cdots \ldots \ldots ..$$

or $\quad N = N_0\left[1 + e^{-\frac{\varepsilon}{kT}} + e^{-\frac{2\varepsilon}{kT}} + e^{-\frac{3\varepsilon}{kT}} + \cdots \dots \dots\right].$

Using the mathematical expression

$$1 + x + x^2 + x^3 + \cdots \dots \dots . = \frac{1}{1-x}$$

and putting $x = e^{-\frac{\varepsilon}{kT}}$,

The preceding equation may be rewritten as:

$$N = \frac{N_0}{1 - e^{-\frac{\varepsilon}{kT}}}$$

Similarly, total energy of the body may be expressed as:

$$E = 0 + \varepsilon . N_0 e^{-\frac{\varepsilon}{kT}} + 2\varepsilon . N_0 e^{-\frac{2\varepsilon}{kT}} + 3\varepsilon . N_0 e^{-\frac{3\varepsilon}{kT}} + \cdots$$

$$E = N_0 \varepsilon e^{-\frac{\varepsilon}{kT}}\left[1 + 2e^{-\frac{\varepsilon}{kT}} + 3e^{-\frac{2\varepsilon}{kT}} + 4e^{-\frac{3\varepsilon}{kT}} + \cdots \dots \dots\right].$$

Using the mathematical formula:

$$1 + 2x + 3x^2 + 4x^3 + \cdots \dots + n . x^{n-1} = \frac{1}{(1-x)^2}$$

Above equation can be written as

$$E = \frac{N_0 \varepsilon e^{-\frac{\varepsilon}{kT}}}{\left(1 - e^{-\frac{\varepsilon}{kT}}\right)^2} = \frac{\dfrac{N_0 \varepsilon e^{-\frac{\varepsilon}{kT}}}{\left(1 - e^{-\frac{\varepsilon}{kT}}\right)^2}}{\dfrac{N_0}{(1 - e^{-\frac{\varepsilon}{kT}})}}$$

Thus we have got expressions for total energy and total number of particles. Substituting these in the expression of average energy we get:

$$\bar{\varepsilon} = \frac{\varepsilon e^{-\frac{\varepsilon}{kT}}}{\left(1 - e^{-\frac{\varepsilon}{kT}}\right)}$$

$$= \frac{\varepsilon}{\left(e^{\frac{\varepsilon}{kT}} - 1\right)}$$

$$= \frac{h\nu}{(e^{\frac{h\nu}{kT}} - 1)}$$

The above equation gives the average energy of oscillator.

It should be noted that the average energy derived from quantum physics differs from the average energy gained from conventional physics, where the average energy per mode is kT.

In the frequency range ν and ν+dν the energy density (total energy per unit volume for a particular frequency range) can be obtained by multiplying the number of Planck's oscillators lying in that particular range multiplied with the average energy of the Planck's oscillator. So we need to calculate the number of oscillators per unit volume lying in the frequency range ν and ν+dν.

Here we can clearly see that the average energy per mode is kT as suggested by Rayleigh Jeans law. Whereas the Planck's quantum radiation law suggests its value equal to $h\nu/(e^{h\nu/kT} - 1)$.

9.12.2 DEDUCTION OF WEIN'S LAW FROM PLANCK'S LAW

We know from the Planck's radiation law

$$E_\lambda d\lambda = \frac{8\pi hc}{\lambda^5}\frac{d\lambda}{e^{he/\lambda kT} - 1}$$

At very small temperature λT is small and for shorter wavelength $e^{hc/\lambda kT}$ becomes large compared to unity and hence Planck's law reduces to

$$E_\lambda\, d\lambda = \frac{8\pi hc}{\lambda^5}\,.\,e^{-hc/\lambda kT}.\ d\lambda$$

This is the required Wein's law, hence Planck's law reduces to Wein's law for shorter wavelengths.

9.12.3 DEDUCTION OF RAYLEIGH-JEANS LAW FROM PLANCK'S LAW

We know from Planck's radiation law:

$E_{\lambda} = \frac{8\pi hc}{\lambda^5} \cdot \frac{d\lambda}{e^{hc/\lambda kT} - 1}$

At very large temperature, λT is large and also for longer wavelengths $e^{hc/\lambda kT}$ can be approximated as $\left(1 + \frac{hc}{\lambda kT}\right)$ (the first two terms of Taylor series expansion $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, -\infty < x < \infty$)

i.e. $$e^{he/\lambda kT} = 1 + \frac{he}{\lambda kT} + \left(\frac{hc}{\lambda kT}\right)^2 \frac{1}{2!} + \ldots = 1 + \frac{hc}{\lambda kT}$$

Hence Planck's law reduces to

$$E_\lambda d\lambda = \frac{8\pi hc}{\lambda^5} \cdot \frac{d\lambda}{(1 + \frac{hc}{\lambda kT} - 1)}$$

$$= \frac{8\pi hc}{\lambda^5} \frac{\lambda kT}{hc} d\lambda$$

$$= \frac{8\pi kT}{\lambda^4} d\lambda$$

This is the required Rayleigh-Jeans law. As a result, at longer wavelengths, Planck's law is reduced to Rayleigh-Jeans law.

Both Wein's law and Rayleigh Jeans law are included into Planck's law.

9.13 SUMMARY

In this unit we have learned about the second law of thermodynamics. The second law clearly explains that it is impossible to convert heat energy to mechanical energy with 100 per cent efficiency. There are two statements on the second law of thermodynamics, and they are Kelvin-Plank statement and Clausius statement. It states that the heat cannot transfer from cold regions to hot regions spontaneously until some external work is done on the system. Kelvin statement states that it is impossible to construct a device operating in a thermodynamic cycle to receive heat from a single heat source and produce a net amount of work.

The concept of an ideal black-body has significance in the study of thermal and electromagnetic radiation energy transmission across all wavelength bands. A black body emits the most radiation relative to any other body at a given temperature. As a result, the black body is utilised as a reference for comparing radiation from real physical bodies. A black body is an ideal body since it absorbs all of the incident radiation and is also an excellent emitter. The study of black body radiation has several uses, including calculating the temperature of far-off stars.

9.14 GLOSSARY

Internal Energy	Energy contained within the system.
Enthalpy	Total heat content of the system.
Entropy	Lack of order or predictability; gradual decline into disorder.
Specific Heat	Amount of heat per unit mass required raising the temperature by 1^0 C.
Heat Capacity	Amount of heat needed to raise the system's temperature by 1^0 C.
Latent Heat	Energy released or absorbed by a thermodynamic system during an isothermal process.
Thermal conductivity	The thermal conductivity of a material is a measure of its ability to conduct heat.
Thermodynamics	Thermodynamics is the study of the relations between heat, work, temperature, and energy

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3. Heat and Thermodynamics, Zemansky and Dittnon.

9.17 TERMINAL QUESTIONS

LONG ANSWER TYPE

1. Give Kelvin-Planck and Clausius statements of the second law and show their equivalence.

2. What do you understand by Carnot Theorem?

3. What is the purpose of the second law of thermodynamics?

4. Define a black-body and write its properties.

5. State and explain Maxwell's equations in detail.

6. Explain Ampere's circuital law. Give its significance. Derive its differential form.

7. Explain Maxwell's correction in Ampere's circuital law.

8. Explain the concept of Maxwell's displacement current and show how it led to the modification of the Ampere's law.

SHORT ANSWER TYPE

1. What are the hypothesis of Planck's law?
2. Derive Planck's radiation law.
3. What are the applications of Maxwell's equation?
4. Define temperature gradient.
5. What is the Maxwell equation?
6. What are the means of heat transfer?
7. What does thermal conductivity depend on?
8. What are the examples of Wien's displacement law?
9. What are the characteristics of Black Body radiations?
10. Define Wien's constant.
11. What is displacement current in Maxwell's equation?
12. Are all four Maxwell's equations independent?
13. What is Black Body radiation?
14. Who derived the expression for Black Body radiation?
15. Define temperature and the scales to measure temperature.

16. What is meant by the term thermal conductivity?
17. What are the factors affecting thermal conductivity?

Multiple Choice based questions:

1. A black body radiation
   a) depends on the temperature of the medium
   b) is function of the temperature of the object
   c) is radiation emmitted by a black body at non-uniform temperature
   d) all of the above

2. The mathematical description of blackbody intensity curve is given by
   a) Wien's law
   b) Planck's law
   c) Rayleigh-Jeans law
   d) Stefan-Boltzmann law

3. The blackbody radiation is
   a) longitudinal wave
   b) electromagnetic wave
   c) sound wave
   d) transverse wave

4. Maxwell's fourth equation is based on ________.
   a) Ohm's law
   b) Ampere's circuital law
   c) Coulomb's law
   d) Faraday's law

5. The Planck's constant h has the dimensions equal to
   a) $M L^{2} T^{-1}$
   b) $M L T^{-1}$
   c) $M L T^{-2}$
   d) $M L T$

6. Maxwell's first equation is based on _________.

a) Gauss's law for magnetism

b) Gauss's law for electrostatic

c) Faraday's law

d) Ampere's circuital law

7. Heat transfer through electromagnetic waves is known as ______.

a) Radiation

b) Conduction

c) Convention

d) None of the options

8. The physical quantity which describes the direction and rate of the temperature change around a particular location is known as ______.

a) Thermal equilibrium

b) Isothermal property

c) Temperature gradient

d) Temperature quotient

9. In Planck's resonators particles can vibrate with

a) only one frequency.

b) frequency of red light.

c) frequencies lies in sound wave range.

d) all frequencies of electromagnetic wave spectrum following quantization of energy.

10. During phase transitions like vaporization, melting and sublimation

a) pressure and temperature remains constant

b) volume and entropy changes

c) both of the mentioned

d) none of the mentioned

11. The value of Planck's constant is

a) $1.6 \times 10^{-27} kg$

b) $1.38 \times 10^{-23} m^2 kg\, s^{-2} K^{-1}$

c) $6.626 \times 10^{-34} Js$

d) $9.1 \times 10^{-31} kg$

12. The average energy of a Planck's oscillator is

a) $-h\upsilon/(1 - e^{\frac{h\upsilon}{kT}})$

b) $h\upsilon/(e^{\frac{h\upsilon}{kT}} - 1)$

c) $h\upsilon/(e^{\frac{1}{kT}} - 1)$

d) $h\upsilon/(1 - e^{\frac{h\upsilon}{kT}})$

13. The relationship at which maximum value of monochromatic emissive power occurs, that is, the relation between the wavelength and temperature of a black body, (Wavelength) MAX T = constant is termed as

a) Planck's law

b) Wein's law

c) Lambert's law

d) Kirchhoff's law

14. Four similar pieces of copper were heated at the same temperature and then left in the environment to cool. Also, these pieces were painted with different colours of paints. Which among the following paints will give fast cooling?

a) White

b) Black

c) Rough

d) Yellow

15. As the wavelength of the radiation declines, the strength of the black body radiations_____

a) Increases

b) Decreases

c) First decreases then increases

d) First increases then decreases

16. The phenomenon in which hot bodies emit radiation is known as?

a) X-rays

b) Black-body radiation

c) Visible light

d) Gamma radiations

17. A black body may not be a perfect emitter of radiations but it's a perfect absorber of radiations.

a) False

b) True

Answer

1. b); 2. b); 3. b),d); 4. b); 5. a); 6. b); 7. a); 8. c); 9. d); 10. c); 11. c); 12. b); 13. b); 14. b); 15. d); 16. b); 17. a)

UNIT 10 GAUSS’S THEOREM, APPLICATIONS

Structure

10.1 INTRODUCTION

The word electrostatics contains two terms electro and statics thereby meaning charge in stationary state. Electrostatics is the study of forces between charges, as described by Coulomb's Law. We develop the concept of an electric field surrounding charges. We know from our early research that the reciprocal interaction of charged entities may be viewed as a result of the force that one exerts on the other, despite the fact that there is no material link between them. This activity was seen uncomfortable and annoying from a distance. Faraday developed the field idea in the nineteenth century to describe the reciprocal interactions of two charged entities. Maxwell later elaborated on this topic. This section will teach you about electric fields, electric field intensity (strength) in various instances, and electric potential.

In this chapter, we will study about the charges, their properties, quantization and conservation of electric charge and Coulomb's law. We will also study electric flux, Gauss's law and the applications of Gauss's law. The various concepts have been presented in a simple and clear manner.

10.2 OBJECTIVES

After studying this unit, you should be able to-

- know about charges and their properties
- learn quantization of charge
- define Coulomb's law
- know about Coulomb's law and their applications in daily life
- learn about electric field, potential and dipoles
- understand electric flux
- understand Gauss's law and its applications

10.3 COULOMB'S LAW

Coulomb's law describes the strength of the electrostatic force (attraction or repulsion) between two charged objects. It is defined as a mathematical concept that defines the electric force between charged objects. Columb's Law states that the force between any two charged particles is directly proportional to the product of the charge but is inversely proportional to the square of the distance between them.

History of Coulomb's Law

Charles Augustin de Coulomb a French mathematician in 1785 first describes a force between two charged bodies in mathematical equations. He stated that the charge bodies repel or attract each other accordingly based on their charge, i.e. opposite charge attracts each other and similar charge repels. He also states the mathematical formula for the force between them, which is called Columb's Law.

Coulomb's Law Formula

Let us suppose that two charges, q_1 and q_2. The force of attraction/repulsion between the charges is represented by the letter 'F,' while the distance between them is represented by the letter 'r.' Then Coulomb's law is mathematically represented as-

F is proportional to the product of the magnitudes of the in-contact charges, i.e. $F \propto q_1q_2$.

F is inversely proportional to the square of the distance between the two in contact charges,

$$F \propto 1/r^2$$

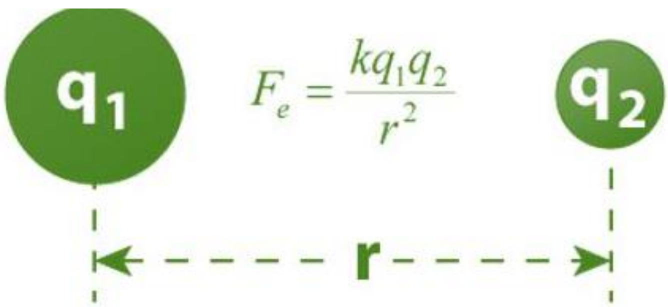

Figure 10.1. Pictorial Representation for Coulomb's Law.

Let's put the two together as follows:

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{q1q2}{r^2}\hat{r}$$

The permittivity of empty space is defined as the constant ε_0. In SI units, where force is measured in Newtons (N), distance is measured in metres (m), and charge is measured in coulombs (C),

$$\varepsilon_0 = 8.85 \times 10^{-12}\ C^2/N.m^2$$

Comparison of Coulomb's Force and Gravitational Force

Along with Coulomb's force, gravitational force also acts between two charged bodies. The comparison between Coulomb's force and Gravitational force is shown below-

S. No.	Coulomb's force	Gravitational force
1.	The Coulomb's force (electrical force) between two charged bodies of charges q_1 and q_2 at separation r is given as- $F_e = \frac{1}{4\pi\varepsilon_0 K}\frac{q_1 q_2}{r^2}$	The gravitational force acting between two bodies of masses m_1 and m_2 at separation r is given as- $F_g = \frac{Gm_1m_2}{r^2}$, where G is known as Universal Gravitational Constant and G = 6.67×10^{-11} N-m^2/Kg^2
2.	The Coulomb force may be attractive or repulsive in nature.	The gravitational force is always attractive in nature.
3.	The Coulomb force depends upon the medium between the charges.	The gravitational force is independent of medium between the masses.
4.	The Coulomb force is much stronger.	The gravitational force is much weaker than the Coulomb force.

Applications of Coulomb's Law

Coulomb's Law is a fundamental physical law. It is used for a variety of reasons, some of which are listed here.

- It calculates the distance and force between two charges.
- It is utilised to keep the charges in a constant equilibrium.
- The electric field is calculated using Columbus' law.

Electric field is given by,

E = F / Q

unit (N/C)

where,

E is the Strength of the electric field

F is the Electrostatic force

Q is the Test charge measured in coulombs

Limitations of Coulomb's Law

Coulomb's Law has certain restrictions, which are discussed further down in the article.

- Coulomb's Law applies to point charges that are at rest only.
- Coulomb's Law is only relevant when the inverse square law is applied.
- Coulomb's Law is only applicable to charges that are considered spherical.
- Coulomb's Law does not apply to charges with arbitrary forms since we cannot know the distance between the charges.

10.4 ELECTRIC CHARGE AND ELECTRIC FIELD

Electric charge

The term "electricity" is derived from the Greek word "Elektron," which meaning "amber." The magnetic and electric forces present in materials, atoms, and molecules affect their properties. The word "electric charge" refers to just two kinds of entities. An experiment revealed two types of electrification: similar charges repelling one another and unlike charges attracting one another. The polarity of charge distinguishes between these two types of charges.

An investigation on frictional electricity-generated electric charges demonstrated that conductors help in the transmission of electric charge whereas insulators do not. Metals, the Earth, and human bodies are all conductors, but porcelain, nylon, and wood are all insulators, resisting the flow of electricity through them significantly.

Basic properties of electric charge

We have seen that there are two kinds of charges, positive and negative, and that their effects tend to cancel each other out. We will now go over some of the additional features of an electric charge. When the sizes of charged bodies are exceedingly small in comparison to the distances between them, they are referred to as point charges. All of the body's charge content is supposed to be concentrated at one place in space.

- **Additivity of charges:** If a system has two point charges, q1 and q2, the total charge of the system is calculated simply by adding q1 and q2, i.e., charges add up like real numbers or are scalars like a body's mass. If a system has n charges q1, q2, q3,..., qn, then the total charge is q1 + q2 + q3 +... + qn. Charge, like mass, has magnitude but no direction. There is, however, one distinction between mass and charge. A body's mass is always positive, but a charge might be positive or negative.
- **Charge is conserved:** It is not possible to create or destroy net charge carried by any isolated system although the charge carrying particles may be created or destroyed in a process.
- **Quantisation of charge:** Experimentally it is established that all free charges are integral multiples of a basic unit of charge denoted by e. Thus charge q on a body is always given by q = ne,

  where n is any integer, positive or negative. The fact that electric charge is always an integral multiple of e is termed as quantisation of charge.

Electric Field

An electric charge generates an electric field, which is an area of space surrounding an electrically charged particle or object in which the charge feels compelled. The electric field exists in all directions in space and may be studied by introducing another charge into it.

An Electric field can be considered an electric property associated with each point in the space where a charge is present in any form. An electric field is also described as the electric force per unit charge. Electric fields are usually caused by varying magnetic fields or electric charges. Electric field strength is measured in the SI unit volt per metre (V/m).

A vector field that may be associated with each point in space, the force per unit charge exerted on a positive test charge at rest at that place, is described mathematically as an electric field.

The electric field formula is as follows:

$$E = \frac{F}{Q}$$

Where,

E is the electric field.

F is the force.

Q is the charge.

The direction of the field is assumed to be the direction of the force acting on the positive charge. The electric field extends radially from the positive charge and in the opposite direction from the negative point charge.

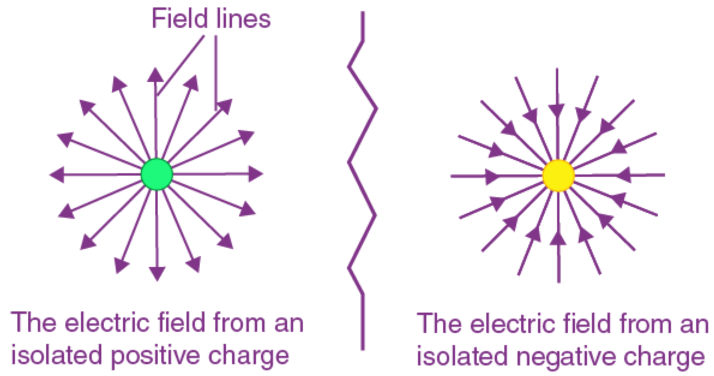

Figure 10.2. Representation of Electric field lines.

The electric charge or time-varying magnetic fields create the electric field. On an atomic scale, the electric field is responsible for the attractive forces that hold the atomic nucleus and electrons together.

Properties of Electric Field

The following are some of the fundamental features of an electric field:

- Due to the negative source, the electric field is always directed towards the charge at any position.
- When a positive source exists, the electric field always points away from the charge.
- If the given charge is positive, the force will have the same direction as the electric field.
- If the given charge is negative, the force will be directed in the opposite direction as the electric field.

The electric field is the sum of positive and negative charges. The electric field is created when these charges exert a force on their surroundings. These electric fields can have either an attracting or repulsive effect. As an example: An electrically charged glass rod draws items such as scraps of paper as it comes into touch with them. After being electrically charged, the rod develops the property of attracting. The physical importance of the electric field is investigated under two main situations: static conditions and electromagnetic non-static settings. The electric field's property is determined by its charges.

10.5 ELECTRIC FIELD INTENSITY

In order to determine the intensity (strength) of electric field at a point in the electric field, let us place an infinitesimal positive test charge q_0 at that point. The force acting on this test charge is measured, and this force divided by the test charge yields the electric field strength. The test charge is considered to be so little that it causes no change in the original electric field. As a result, the electric field strength (or intensity) is defined as follows:

"The intensity of electric field at a point in an electric field is the ratio of the force acting on the test charge placed at that point to the magnitude of the test charge". It is a vector quantity and its direction is along the direction of force.

Thus, if F is the force acting on a test charge q_0 at a location in an electric field, the strength of the electric field E at that point is given by-

$$\vec{E} = \frac{\vec{F}}{q_0}$$

Assuming that the test charge q_0 is infinitesimal, the definition of electric field intensity may be stated as-

$$\vec{E} = \lim_{q_0 \to 0} \frac{\vec{F}}{q_0}$$

The force F is a vector quantity, whereas the test charge q_0 is a scalar quantity. As a result, the intensity of the electric field dE will be a vector quantity with the same direction as the direction of the force F, i.e. the direction in which the positive charge put in the electric field tends to move. If the test charge is negative, the direction of the electric field E will be the opposite of the force acting on the negative charge.

Obviously, the unit of intensity (strength) of an electric field is Newton/metre.

If we know the intensity of an electric field E at a given point in an electric field, we can use the following equation to calculate the force F acting on a charge q placed at that point:

F = qE

The intensity (strength) of an electric field is obviously measured in Newton/metre.

If we know the strength of an electric field E at a particular site in an electric field, we can use the equation below to compute the force F acting on a charge q placed at that location:

10.6 ELECTRIC LINES OF FORCE

We studied the electrostatic force experienced by a charge in an electric field. If the charge is free, it will move in the opposite direction of the force. If the direction of the force varies continually, the direction of motion of the charge likewise changes continuously, i.e. it goes along a curved route. The 'electric line of force' is the route of a free positive charge in an electric field. As a result, "an electric line of force is a smooth imaginary curve drawn in an electric field along which a free, isolated unit positive charge moves." The direction of the force operating on a positive charge deposited at any point on the electric line of force is given by the tangent formed at that location."

The intensity (strength) of an electric field may now be defined in terms of electric lines of force as follows-

“The intensity of electric field at any point is defined as a vector quantity whose magnitude is measured by the number of electric lines of force passing normally through per unit small area around that point and whose direction is along the tangent on line of force drawn at that point”.

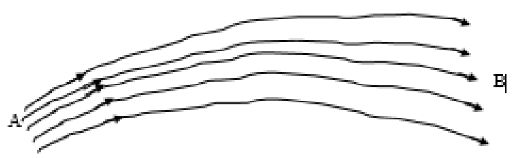

Figure 10.3: Electric lines of force.

10.6.1 Properties of Electric Lines of Force

- The lines of force begin with a positive charge and conclude with a negative charge.
- Electric field lines are never closed loops.
- They are always perpendicular to the charge's surface.
- The lines of force are never crossed.

- Electric force lines do not travel through the conductor.
- The relative proximity of lines of force in distinct regions of space indicates the respective intensities of the electric field in those places.
- Electric force lines contract lengthwise.
- Force lines impose lateral pressure on one another.

10.7 PHYSICAL SIGNIFICANCE OF ELECTRIC FIELD

The electric field is a vector quantity which may vary from point to point in magnitude and direction. The magnitude of electric field at any point is a measure of electric force on a unit positive test charge, assuming that the test charge does not perturb the field of the system and its direction is that of electrostatic force on the test charge. This implies that the electric field is the characteristic of the charges of system and is independent of the test charge. The test charge is simply introduced for measurement of electric field in a suitable manner.

The true physical significance of electric field appears only when we keep in view that electrostatic interaction is only a part of general fundamental force known as electromagnetic interaction. When two charges q_1 and q_2 are in accelerated motion, then either accelerated charge (say q_1) produces electromagnetic wave which propagates with speed of light; reaches on another charge (say q_2) and causes a force on it.

Thus, the force between two distant charges is not instantaneous but appears with a time delay. Thus electric field (as well as magnetic field) is detected by their interaction forces; but they are not simply mathematical terms but are regarded as physical quantities which may be measured by the forces exerted by them on single charges or diploes.

10.8 ELECTRIC POTENTIAL

The electric field produced by a charge may be represented in two ways:

I. by the intensity of the electric field E at a point in the field and
II. by the electric potential V.

The strength of the electric field E is a vector quantity, whereas V is a scalar quantity. Both of these variables are linked. The electric potential is a crucial parameter in the study of electric fields. They are both distinctive features of a point in space.

We know that a free positive charge in an electric field tends to flow in the direction of the electric field. Work is done against the Coulomb's force of repulsion when a positive test charge is introduced in the opposite direction of the electric field. The potential at infinity is assumed to be 0 in order to establish absolute potential at any place.

“The electric potential at any point in an electric field is defined as the work done by external force in carrying unit positive test charge from infinity to that point, without any acceleration”.

If W represents the work done in moving a positive test charge q0 from infinity to any location in an electric field, then the electric potential at that point is-

$V = \frac{W}{q_0}$

The electric potential is a scalar quantity. Joule/Coulomb is its SI unit. Volt is another unit.

If q_0 =1 coulomb, W= 1 Joule, then

$V = \frac{1 \text{ Joule}}{1 \text{ Coulomb}} = 1$ volt

1 volt is the electric potential at a location in an electric field if the work done in carrying one coulomb of electric charge from infinity to that point is one joule, provided the charge of one coulomb has no effect on the initial electric field.

10.8.1 Potential Difference

Potential difference between any two points in the presence of the electric field is defined as the amount of work done in moving a unit positive charge without acceleration from one point to another along any path between the two points. It is given as (dV):

$$dV = \frac{dw}{dq}$$

The potential difference between points A and B, on the other hand, is defined as the shift in the potential energy of a charge q divided by the charge changed from A to B.

$V_A - V_B = \frac{W_{BA}}{q_0}$

Because the work W_{BA} and the charge q_0 are both scalar quantities, the potential difference $V_A - V_B$ is likewise a scalar number. The unit of work done W_{BA} is Joule and the unit of charge q_0 is coulomb. Therefore, the unit of potential difference is Joule/Coulomb.

Now we can define 1 volt potential difference. If W_{BA}= 1 Joule, q_0 = 1 Coulomb then

$V_A - V_B = \frac{1\ \text{Joule}}{1\ \text{Coulomb}} = 1$ volt

i.e. if 1 joule of work is done in carrying a test charge of 1 Coulomb from one point to the other in an electric field, then the potential difference between those points will be 1 volt.

10.8.2 Physical Significance of Electric Potential

Positive charge always flows from greater potential to lower potential, just as liquid flows from higher pressure (or higher level) to lower pressure (or lower level) and heat flows from higher temperature to lower temperature. There is no relationship between the direction of charge flow and the quantity of charge, as there is in the case of liquid or heat flow. Thus, electric potential is the physical quantity that governs the direction of positive charge flow. When two conducting bodies with uneven potentials come into touch, the charge continues to flow from one to the other until their potentials equalize. Positive charge always flows from greater potential to lower potential, and negative charge always flows from lower to higher potential. When two conductors are kept in touch, electrons move from lower to higher potential until their potentials become equal.

10.9 ELECTRIC FLUX

Electric flux is the property of an electric field that quantitatively explains the number of electric lines of force (or electric field lines) that intersect with the given area. Electric field lines are considered to originate on positive electric charges and terminate on negative charges. In simple words, the total number of electric field lines passing a given area in a unit of time is defined as the electric flux. The SI base unit of electric flux is voltmeters (V m). Let us imagine the flow of water with a velocity *v* in a pipe in a fixed direction, say to the right. If we take the cross-sectional plane of the pipe and consider a small unit area given by *ds* from that plane, the volumetric flow of the liquid crossing that plane normal to the flow can be given as *vds*. When the plane is not normal to the flow of the fluid but is inclined at an angle θ, the total volume of liquid crossing the plane per unit time is given as $vds.\cos\theta$. Here, $ds\cos\theta$ is the projected area in the plane perpendicular to the flow of the liquid. Similar to the example above, if the plane is normal to the flow of the electric field, the total flux is given as:

$$\Phi = EA$$

When the same plane is tilted at an angle θ, the projected area is given as $A\cos\theta$, and the total flux through this surface is given as:

$$\Phi = EA\cos\theta$$

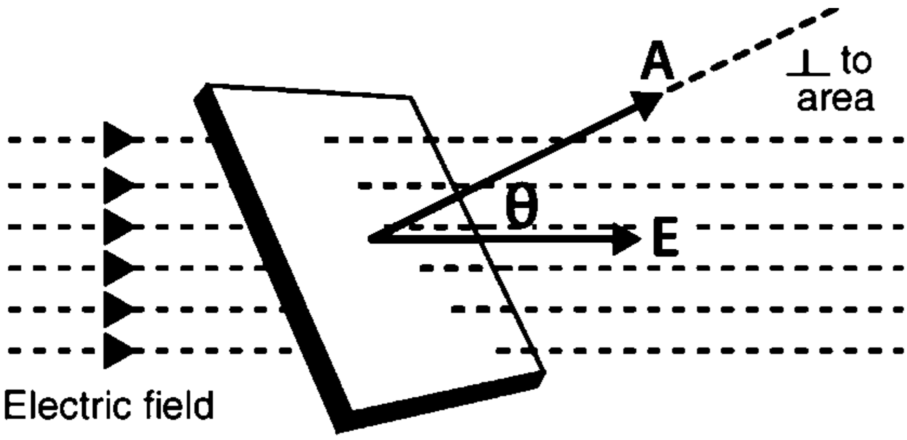

Figure 10.4. Direction of the electric flux.

Electric flux is a scalar quantity. Its unit is Newton-metre^{2}Coulomb^{-1}.

10.10 GAUSS'S THEOREM

The mathematical relationship between the charge enclosed in a material and electric flux is defined by the Gauss theorem. **Gauss's law** states that the total electric flux emerging out of a closed surface is equal to **$1/\varepsilon_0$ times the charge enclosed by the closed surface, where ε_0 is the permittivity of the free space.** By Gauss law, it can be said that if no charges are enclosed by a surface, then the net electric flux remains zero. Gauss's law is considered true for any closed surface, despite the shape or size.

Mathematically,

$$\Phi = \frac{Q}{\varepsilon o}$$

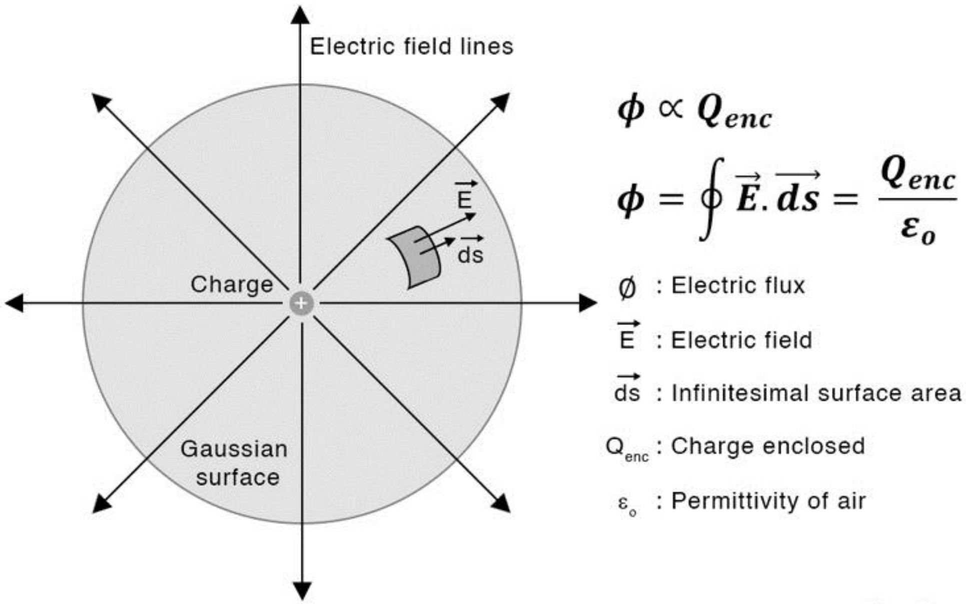

Figure 10.5. Illustration of the Gauss law.

The electric flux from any closed surface originates due to the positive charges and terminates at the negative charges of the electric field enclosed by the surface. The charges existing outside the surface do not contribute to the total electric flux. The applications of Gauss Law are mainly to find the electric field due to infinite symmetries such as:

- Uniformly charged Straight wire.
- Uniformly charged Infinite plate sheet.
- Uniformly charged thin spherical shell.

10.11 APPLICATIONS OF GAUSS'S THEOREM

It is quite interesting that Gauss' theorem gives a straightforward approach for determining the strength of an electric field in symmetrical instances. Consider an imaginary Gaussian surface symmetrical to a given charge, compute electric flux through it, and equate this flux to the $1/\varepsilon_0$ charge encompassed by the surface. Let us now look at several interesting applications of Gauss' theorem.

10.11.1 Electric Field due to a Point-charge

Every charge in the universe exerts a force on every other charge in the universe" is a bold yet true statement of physics. One way to understand the ability of a charge to influence other charges anywhere in space is by imagining the influence of the charge as a field. Note that this 'influence' is simply the electrostatic force that a charge is able to exert over another. However, when describing fields, we require a quantity (scalar or vector), that is independent of the charge it is acting on and only dependent on the influence and the spatial distribution.

So, in a simple way, we can define the electrostatic field considering the force exerted by a point charge on a unit charge. In other words, we can define the electric field as the force per unit charge.

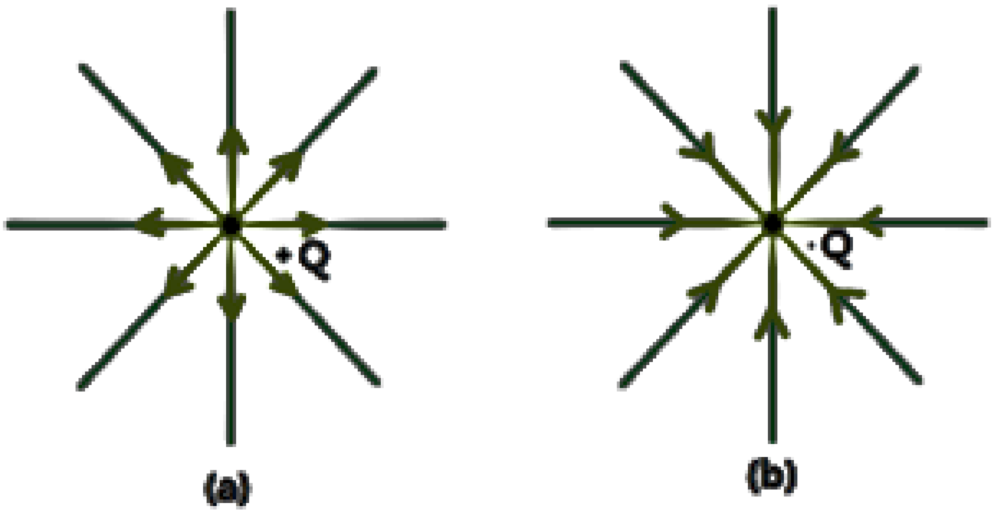

Figure 10.6. Nature of electric field lines of positive and negative charges.

When a glass rod is rubbed with a piece of silk, it acquires the property of attracting objects like pieces of paper, towards it. This happens due to the discharge of electric charges by rubbing of insulating surfaces. Electric charge is a property that accompanies fundamental particles, wherever they exist. When an electric charge q_0 is held in the vicinity of another charge Q, q_0 either experiences a force of attraction or repulsion. We say that this force is set up due to the electric field around the charge Q. Therefore, we can say that the electric field of charge Q acts as a space by virtue of which the presence of charge Q modifies the space around itself leading to the generation of force F on any charge q_0 held in this space. The concept of the field was first introduced by Faraday and the force that occurred due to this field is mathematically expressed as,

$$\vec{F} = \frac{kQq_1}{r^2}$$

Where r is the distance between the test charge and the source charge, q1 is the test charge, Q is the source charge and k is the constant which is expressed as:

$$k = \frac{1}{4\pi\epsilon 0} = 9\times10^9 \text{ Nm}^2 \text{ C}^{-1}$$

The electric field due to a given electric charge Q is defined as the space around the charge in which electrostatic force of attraction or repulsion due to the charge Q can be experienced by another test charge q.

The electric field intensity at any point is the strength of the electric field at that point. It is defined as the force experienced by a unit positive charge placed at a particular point. Here, if force acting on this unit positive charge $+q_0$ at a point r, then electric field intensity is given by:

$$\vec{E}(r) = \frac{\vec{F}(r)}{q_o}$$

Hence, E is a vector quantity and is in the direction of the force and along the direction in which the test charge +q tends to move. Its unit is NC^{-1}.

10.11.2 Electric Field due to a charged spherical shell

Let us consider a charged spherical shell. To find an electric field outside the spherical shell, we take a point P outside the shell at a distance r from the center of the spherical shell. A Gaussian spherical surface of radius r and center O. To determine the electric field due to a uniformly charged thin spherical shell is possible to obtain with the help of Gauss's law. In this article, let's learn about Electric fields due to spherical shells at the surface, inside and outside by using Gauss law. To determine the electric field due to a uniformly charged thin spherical shell, the following three cases are considered:

Case 1: At a point outside the spherical shell where r > R.

Case 2: At a point on the surface of a spherical shell where r = R.

Case 3: At a point inside the spherical shell where r < R.

Let us consider each case separately to determine the electric field.

Case 1: At a point outside the spherical shell where r > R.

Let P be the point outside the shell at a distance r from the center. Since the surface of the sphere is spherically symmetric, the charge is distributed uniformly throughout the surface. A spherical Gaussian surface with the radius r and total charge enclosed on this Gaussian surface Q is selected. If Q > 0, then the electric field is radially pointed outward and if Q < 0, then the electric field is radially pointed inward.

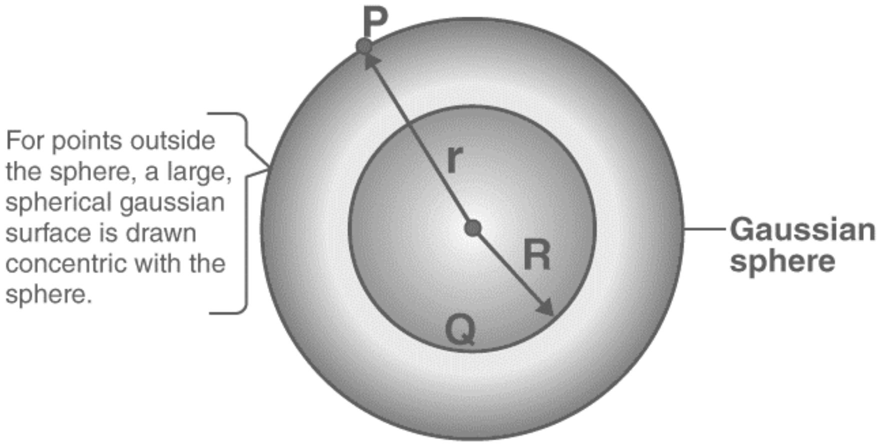

Figure 10.7. Illustration of the Gaussian surface for *r* > *R shell*.

From Gauss law, we know that:

$$\oint \vec{E}.d\vec{A} = \frac{Q}{\epsilon_0}$$

$$\vec{E} \oint gaussian\ surface\ \ dA = \frac{Q}{\epsilon_0}$$

$$\vec{E}\ (4\pi r^2) = \frac{Q}{\epsilon_0}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}$$

We can say that the electric field at a point outside the shell will remain the same if the entire charge Q is concentrated at the center of the spherical shell.

Case 2: At a point on the surface of a spherical shell where r = R.

Let P be the point at the surface of the shell at a distance r from the center. In this case, r = R; since the surface of the sphere is spherically symmetric; the charge is distributed uniformly throughout the surface. From Gauss law, we know that,

$$\oint \vec{E}.d\vec{A} = \frac{Q}{\epsilon_0}$$

$$\vec{E} \oint gaussian\ surface\ \ dA = \frac{Q}{\epsilon_0}$$

$$\vec{E}\ (4\pi r^2) = \frac{Q}{\epsilon_0}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}\hat{r}$$

Case 3: At a point inside the spherical shell where r < R.

Let P be the point inside the spherical shell at a distance r from the centre. In this case, r < R. From Gauss law, we know that

$$\oint \vec{E}.d\vec{A} = \frac{Q}{\epsilon_0}$$

$$\vec{E} \oint gaussian\ surface\ \ dA = \frac{Q}{\epsilon_0}$$

$$\vec{E}\ (4\pi r^2) = \frac{Q}{\epsilon_0}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}$$

We know that the Gaussian surface does not enclose any charge, therefore, Q = 0.

Thus, E = 0

Hence, we can say that the electric field due to the uniformly charged thin spherical shell is zero at all the points inside the shell.

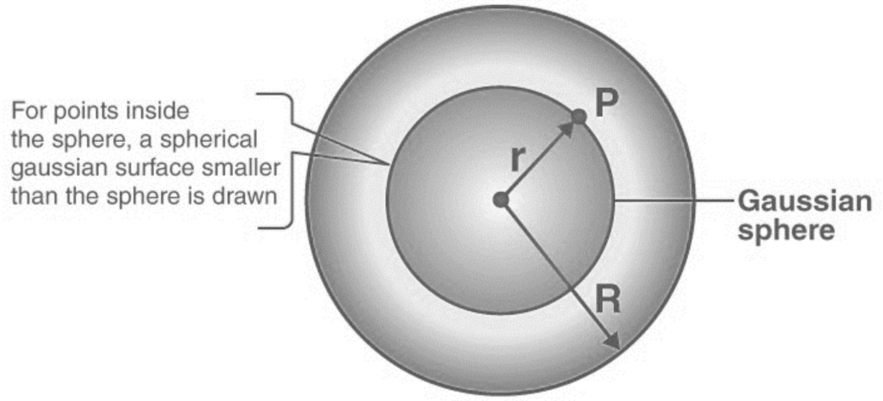

Figure 10.8. Illustration of the Gaussian surface for *r* < *R shell*.

10.12 SUMMARY

In this unit, you have learned about electric charge, how it was found, and its properties. You have learned that motion has no influence on a body's charge, i.e. the charge on a body or particle is constant whether it is at rest or traveling at any velocity. Charge is conserved, which means it cannot be generated or destroyed but can only be moved from one body to another. You have also studied charge quantization, which states that electric charge cannot be split endlessly. Coulomb's law, its validity requirements, and its significance is also discussed. "The force of attraction or repulsion between two point charges is directly proportional to the product of the charges and inversely proportional to the square of the distance between them," according to Coulomb's law. This force is directed along the line connecting the two charges." Concept of electric field intensity, electric potential, electric field lines, potential differences are also studied in detail. You have also learned about the Gauss Law and its application to find the electric field due to infinite symmetries.

To check the progress, self-assessment questions (SAQs) are given place to place.

10.13 GLOSSARY

Conserved- preserved

Electric force- the force experienced by a charge placed at a point in an electric field

Electric flux - the measure of flow of the electric field through a given area.

Significance – importance, noteworthiness, a concealed or real meaning.

Quantization - the process of mapping continuous infinite values to a smaller set of discrete finite values.

Electric Field - a region around a charged particle or object within which a force would be exerted on other charged particles or objects.

Electric potential - the amount of work needed to move a unit charge from a reference point to a specific point against an electric field.

Potential difference - the difference of electrical potential between two points.

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10.16 TERMINAL QUESTIONS

1. What is an electric field?
2. When is the electric field said to be uniform?
3. Explain quantization of charge. Hence define elementary charge.

4. When is the electric field said to be non-uniform?
5. Does a charge experience a force due to its own field?
6. Does Coulomb's law of electric force obey Newton's third law of motion?
7. Give the importance of Coulomb's law.
8. Give comparison of Coulomb's force and Gravitational force.
9. What do you mean by electric flux? What is its unit?
10. Write down the significance of electric flux.
11. State the Gauss's theorem in electrostatics.
12. Proof Gauss's theorem in electrostatics
13. Why do two electric field lines never intersect?
14. The test charge used to measure the electric field intensity at a point should be infinitesimally small. Why?
15. Is electric field intensity a scalar or a vector quantity?
16. Define electric intensity at a point in an electric field.
17. Write any two properties of electric field lines.
18. Define the potential difference between two points.
19. Explain the statement the potential difference between two points is 1 volt'
20. Why is a series arrangement not used for connecting domestic electrical appliances in a circuit?

Multiple Choice Based Questions:

1. Electric Field intensity is a….

a) Dimensionless quantity

b) vector quantity

c) scalar quantity

d) all of the above

2. Potential difference between two points is equal to

a) electric charge /time

b) work done/time

c) work done/charge

d) work done X charge

3. Gauss law is related to _____ fluxes?

a) Electric

b) Magnetic

c) Chemical

d) Both a and b

4. Gauss law is applicable for ___ type of surface?

a) Closed

b) Open

c) Feedback

d) Both a and c

5. Gauss law is related to _____ law?

a) Coulombs law

b) Charles law

c) Ohms law

d) Farads law

6. Electric potential difference between two points in an electric circuit carrying some current is the _____ done to move a ______ charge from one point to the other.

a) Force, small

b) Work, Large

c) Mass, unit

d) Work, unit

7. The work done in moving a unit charge across two points in an electric circuit is a measure of:

(a) current

(b) potential difference

(c) resistance

(d) power

8. Two electric field lines ______

a) Always intersect each other

b) Never intersect

c) May intersect sometimes

d) Are always perpendicular to each other

9. Which among the following is false about electric field lines?

a) They are continuous

b) They attract each other

c) They remain parallel in a uniform electric field

d) They diverge from positive charge

10. The electric field lines passing through a certain area is known as:

a) Electric flux

b) Electric field

c) Electrostatics

d) Electric field lines

11. Flux will be minimum when the electric field lines are:

a) Parallel to area the vector.

b) Perpendicular to the area vector.

c) At an acute angle to the area vector.

d) None of the above.

12. System International unit of electric flux is:

a) Nm^2C^{-1}

b) NmC

c) Nm^2C

d) N^2mC

13. If there existed only one type of charge q on the Earth, then what is the electric flux related to Earth?

a) Zero through any surface of Earth.

b) Infinite flux on Earth.

c) Zero if the charge is placed outside Earth and q/ε0 if the charge is placed inside the earth.

d) Cannot be defined.

True/False

14. Electric potential difference between two points is the work done to move a unit charge from one point to the other.

a) True

b) False

15. Work done in moving a unit positive charge from one point to other in an electric circuit is called potential difference.

a) True

b) False

10.17 ANSWERS

1: b), 2: c), 3: d), 4: a), 5: a), 6: d), 7: b), 8: b), 9: b), 10: a), 11: b), 12: a), 13: c), 14: a), 15: a).

UNIT 11 MAGNETISM

Structure

11.1 INTRODUCTION

The magnetic effects can be produced by a magnet or by a current carrying conductor. The region around a magnet or current carrying conductor, in which a magnetic needle experiences a torque and rests in a definite direction, is called 'magnetic field'. A charge moving in a magnetic field experiences a deflecting force. Of course, if a charge moving through a point experiences a deflecting force, then a magnetic field is said to exist at that point. This field is represented by a vector quantity $\vec{B}$, called magnetic field or magnetic induction. The magnetic induction can be defined in terms of lines of induction as the number of lines of induction passing through a unit area placed normal to the lines measures the magnitude of magnetic induction or magnetic flux density $\vec{B}$. Obviously, in a region smaller is the relative spacing of the lines of induction, the greater is the magnetic induction. The tangent to the line of induction at any point gives the direction of magnetic induction $\vec{B}$ at that point. The lines of induction simply represent graphically how $\vec{B}$ varies throughout a certain region of space. In the present unit, you will study the force on a moving charge in simultaneous electric and magnetic fields, Biot-Savart law, magnetic force between current elements, Ampere's circuital law and its applications.

11.2 OBJECTIVES

After studying this unit, you should be able to-

- understand Lorentz force
- apply Biot-Savart law
- apply Ampere's circuital law
- solve problems using Biot-Savart law and Ampere's circuital law

11.3 LORENTZ FORCE

Let us consider a charged particle of charge q which is moving with velocity $\vec{v}$ in a magnetic field $\vec{B}$, then the magnetic force acting on that charged particle is given by -

$$\vec{F} = q(\vec{v} \times \vec{B}) \quad \text{.....(1)}$$

The direction of $\vec{F}$ will be perpendicular to both the direction of velocity $\vec{v}$ and the direction of magnetic field $\vec{B}$. Its exact direction is given by the law of vector product of two vectors.

The magnitude of magnetic force is given as-

$$F = qvB \sin\theta \quad \text{.....(2)}$$

where θ is the angle between velocity $\vec{v}$ and magnetic field $\vec{B}$.

$\vec{v}$

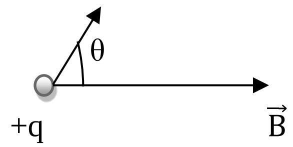

Figure 1

If the angle between velocity $\vec{v}$ and magnetic field $\vec{B}$ is 90^0 then-

$F_{max} = qvB \sin 90^0 = qvB$

i.e. if velocity $\vec{v}$ and magnetic field $\vec{B}$ are at right angle then the magnetic force acting on the charged particle is maximum that is equal to qvB.

If $\theta = 0^0$ or 180^0 i.e. velocity $\vec{v}$ and magnetic field $\vec{B}$ are parallel to each other then-

$F = qvB \sin 0^0 = 0$

i.e. if the charged particle is moving parallel to the magnetic field, it does not experience any force.

If v = 0, then F = 0. This means that if the charged particle is at rest in the magnetic field, then it does not experience any force.

If a charged particle is moving in space where both an electric field $\vec{E}$ and a magnetic field $\vec{B}$ are present, then the total force acting on the charged particle is called the Lorentz force.

The electric force acting on charged particle, $\vec{F}_e = q\vec{E}$ .....(3)

The magnetic force acting on the charged particle, $\vec{F}_m = q(\vec{v} \times \vec{B})$

The total force acting on the charged particle, $\vec{F} = \vec{F}_e + \vec{F}_m$

$= q\vec{E} + q(\vec{v} \times \vec{B})$

or $\vec{F} = q[\vec{E} + (\vec{v} \times \vec{B})]$ .....(4)

The force given by equation (4) is called the Lorentz force and the equation is known as Lorentz force equation.

If a charged particle enters perpendicular to both the electric and magnetic fields, then it may cancel each other and therefore, the charged particle will pass undeflected. In this situation,

$\vec{F} = q[\vec{E} + (\vec{v} \times \vec{B})] = 0$

or $\vec{E} = -(\vec{v} \times \vec{B})$ .....(5)

In magnitude, $E = v \times B$ or $v = \frac{E}{B}$ .....(6)

Thus a charged particle entering in simultaneous electric and magnetic field may pass undeflected. Such an arrangement of simultaneous electric and magnetic fields is called velocity-selector. Because the charged particle of only specified velocity given by v = E/B can pass undeflected. The particle of velocity v < E/B will be deflected towards electric force and those with velocity v > E/B will be deflected towards magnetic force.

11.4 MAGNETIC INDUCTION AND BIOT-SAVART LAW

Magnetic induction or magnetic flux density describes magnetic field at a point near a magnet or current carrying conductor in space. The magnetic induction is a vector quantity and generally denoted by B. The field B is closely related to the magnetic field H, often called the magnetic field intensity, and generally denoted by H. Sometimes, authors refer to B as the magnetic field. Oersted's experimentally showed that a current-carrying conductor produces a magnetic field around it.

French scientists Biot and Savart, in the same year 1820, performed a series of experiments to study the magnetic fields produced by various current-carrying conductors and formulated a law to determine the magnitude and direction of the magnetic fields so produced. This law is known as 'Biot-Savart law'. Let us consider a conductor of an arbitrary shape carrying electric current i and P a point in vacuum at which the magnetic field is to be determined. Let us divide the conductor into infinitesimal current-elements. Let us consider a small current element XY of length dl.

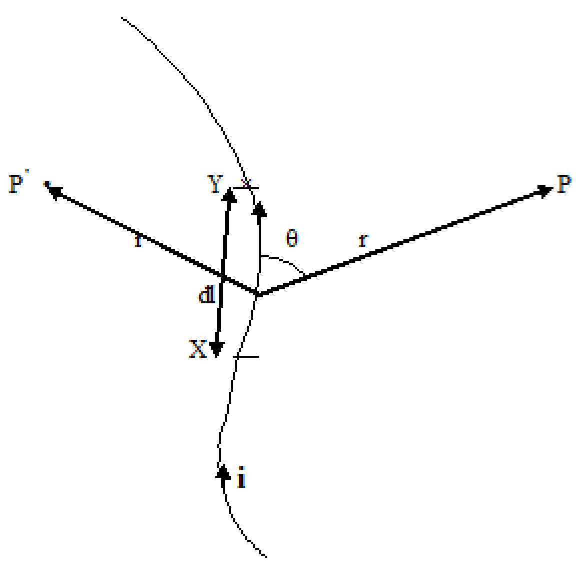

Figure 2: Magnetic field due to Current carring conductor

According to Biot-Savart law, the magnetic field dB produced due to this current element at point P at a distance r from the element is-

(i) directly proportional to the current flowing in the element i.e. $dB \propto i$

(ii) directly proportional to the length of element i.e. $dB \propto dl$

(iii) directly proportional to sin of angle between current element and the line joining current element to point P i.e. $dB \propto \sin\theta$

(iv) inversely proportional to the square of the distance of the element from point P i.e. $dB \propto \frac{1}{r^2}$

Combining these, we get-

$dB \propto \frac{idl\sin\theta}{r^2}$

or $dB = \frac{\mu}{4\pi}\frac{idl\sin\theta}{r^2}$.....(7)

where, $\frac{\mu}{4\pi}$ is a dimensional constant of proportionality whose value depends upon the units used for the various quantities. It depends on the medium between the current element and point of observation (P). Here, μ is called the permeability of medium. Equation (7) is called Biot-Savart law. The product of current i and the length of element dl i.e. idl is called the current element. Current element is a vector quantity; its direction is along the direction of current.

If you place the conductor in vacuum or air, then μ is replaced by $\mu_0$and thus Biot-Savart law can be written as-

$$dB = \frac{\mu_0}{4\pi}\frac{idl\sin\theta}{r^2}.....(8)$$

μ_0 is called the permeability of free space or air.Its value in the SI system is assigned as-

$\mu_0 = 4\pi \times 10^{-7}$ weber/ampere-meter ($WbA^{-1}m^{-1}$)

Thus,$\frac{\mu_0}{4\pi}$= 10^{-7} $WbA^{-1}m^{-1}$

μ_0 or $\frac{\mu_0}{4\pi}$ may also be expressed in Newton/Ampere^2 (N/A^2).

The direction of magnetic field is perpendicular to the plane containing current element and the line joining point of observation to current element. Therefore, in vector form, Biot-Savart law can be expressed as-

$d\vec{B} = \frac{\mu_0}{4\pi}\frac{id\,\vec{l}\times\vec{r}}{r^3}$ .....(9)

The resultant magnetic field at P due to the whole conductor can be found by integrating equation (9) over the entire length of the conductor. Thus

$\vec{B} = \int d\vec{B}$

Direction of magnetic field dB: The direction of magnetic field $d\vec{B}$ is perpendicular to both the current element $id\vec{l}$ and the position vector $\vec{r}$ of point P relative to current element and may be found by the law of vector cross product or by Maxwell's right hand screw rule. Thus in figure 2 the direction of magnetic field at point P is shown by × (cross) i.e. vertically inward (downward perpendicular to the plane of the paper) and at point P', the direction of magnetic field is shown by •(dot) i.e. vertically outward(upward perpendicular to the plane of the paper).

11.4.1 Maxwell's Right Hand Screw Rule:

If we hold the screw driver in our right hand and rotate a screw in such a way that the point of screw moves along the direction of electric current in the conductor, then the direction of rotation of the thumb will be the direction of magnetic lines of force.

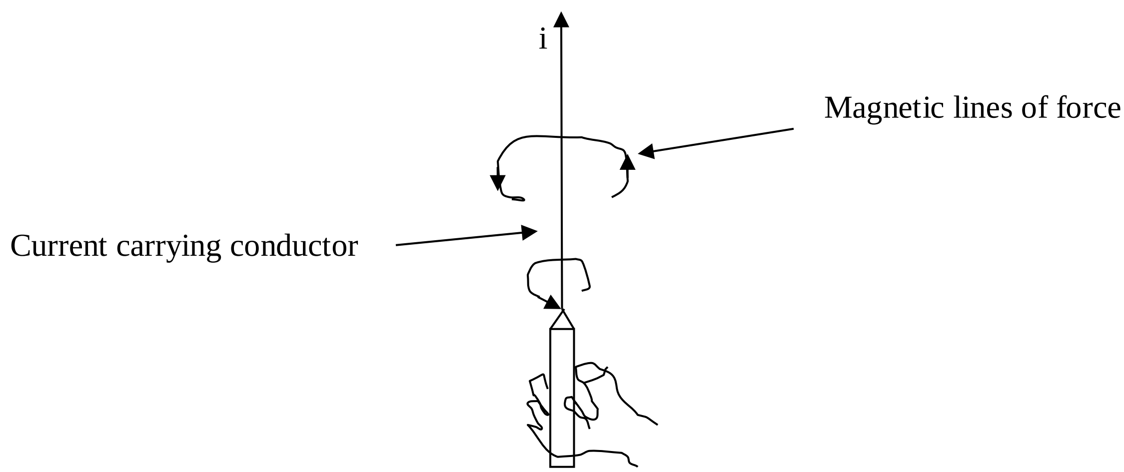

Figure 3

11.4.2 Comparison of Coulomb's Law and Biot-Savart Law

A current generates a magnetic field in the surrounding space while a stationary charge generates an electric field. Coulomb's law gives the electric field due to a distribution of charges while Biot-

Savart law gives the magnetic field due to a current element. According to Coulomb's law, the magnitude of electric field at a point distant r due to a charge element dq is given as-

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}$$

According to Biot-Savart law, the magnitude of magnetic field at a point distant r due to a current element i dl is given as-

$$dB = \frac{\mu_0}{4\pi}\frac{idl\sin\theta}{r^2}$$

where θ is the angle between the length of the element and the line joining the element to the point.

We, thus, see that Biot-Savart law is the magnetic equivalent of Coulomb's law and both are inverse square laws. However, these two laws differ in certain respect. The charge element dq is a scalar while the current element i dl is a vector ($i d\vec{l}$) whose direction is in the direction of the current. According to Coulomb's law, the magnitude of electric field depends only upon the distance of the charge element from the point. According to Biot-Savart law, the magnitude of magnetic field at the point also depends upon the angle between the current element and the line joining the current element to the point. Secondly, according to Coulomb's law, the direction of electric field is along the line joining the charge element and the point. According to Biot-Savart law, the direction of magnetic field is perpendicular to the current element as well as to the line joining the current element to the point.

Example 1: An electron moving with velocity 5×10^7 m/sec enters a magnetic field of 1 Wb/m^2 at an angle of 90^0 to the magnetic field. Estimate the magnetic force acting on the electron.

Solution: Here v = 5×10^7 m/sec, B = 1 Wb/m^2, θ = 90^0 , q = e = 1.6×10^{-19} C

Using F = qvB sinθ

F = $1.6\times10^{-19}\times5\times10^7\times1\times\sin90^0$

= 8×10^{-12} Newton

Example 2:A proton is moving northwards with a velocity of 3×10^7 m/sec in a uniform magnetic field of 10 Tesla directed eastward. Find the magnitude and direction of the magnetic force on the proton. Charge on proton= 1.6×10^{-19} C

Solution: Given v = 3×10^7 m/sec, B= 10Tesla, q = 1.6×10^{-19} C

The magnetic force on proton F = qvB sinθ

= $1.6\times10^{-19}\times3\times10^7\times10\times\sin 90^0 = 1.6\times10^{-19}\times3\times10^7\times10\times1 = 4.8\times10^{-11}$ Newton

The magnetic field is directed eastward and the direction of motion of proton is northward i.e. the direction of flow of current is northward. By Fleming's left-hand rule, the force on the proton will be directed vertically downwards.

Self Assessment Question (SAQ) 1: An electron is moving vertically upward with a speed of 2×10^8 m/sec. Find out the magnitude and direction of the force on the electron exerted by a horizontal magnetic field of 0.50 Wb/m^2 directed towards west? Also calculate the acceleration of the electron.

Self Assessment Question (SAQ) 2: An electron moving with velocity $\vec{v}$ along +x-axis enters a uniform magnetic field $\vec{B}$ directed along + y-axis. What is the magnitude and direction of the force on the electron?

Self Assessment Question (SAQ) 3: A 2 MeV proton is moving perpendicular to a uniform magnetic field of 2.5 Tesla. Find the force on the proton. The mass of proton = 1.65×10^{-27} Kg.

Self Assessment Question (SAQ) 4: Choose the correct option-

The force on a charged particle moving in a magnetic field is maximum when the angle between direction of motion and field is-

(i) 45^0 (ii) 180^0 (iii) zero (iv) 90^0

Self Assessment Question (SAQ) 5: Choose the correct option-

A moving electric charge produces-

(i) electric field only (ii) magnetic field only (iii) both electric and magnetic fields (iv) neither of these two fields

11.5 MAGNETIC FORCE BETWEEN TWO PARALLEL CURRENT CARRYING CONDUCTORS

Let us consider two long, straight and parallel current carrying conductors PQ and RS separated by a distance r. Let i_1 and i_2 be the currents flowing in two conductors in the same direction respectively. Now, let us find expression for the force acting between the conductors.

The magnitude of the magnetic field at a point P on conductor RS is –

$$B = \frac{\mu_0}{4\pi}\frac{2i_1}{r}$$

By Maxwell's right hand screw rule, the direction of this field is perpendicular to the plane of the page directed downward.

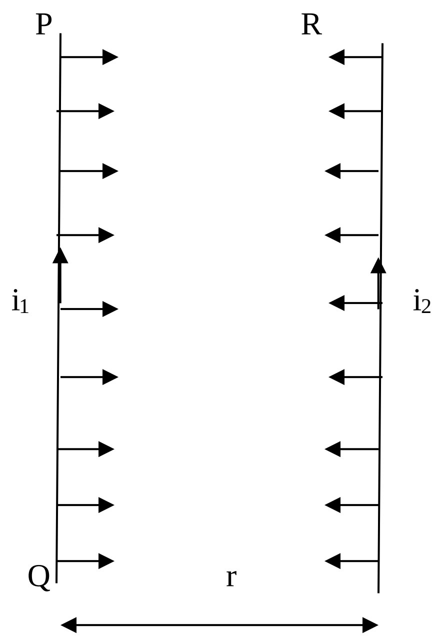

Figure 4

Obviously, the conductor RS is situated in magnetic field B perpendicular to its length. It, therefore, experiences a magnetic force. Using formula, F = iBl sinθ, the magnitude of magnetic force acting on a length *l* of conductor RS is given as-

$F = i_2 B l \sin\theta = i_2 \frac{\mu_0}{4\pi}\frac{2i_1}{r} l \sin 90^0$

Or $$F = \frac{\mu_0}{4\pi}\frac{2i_1i_2l}{r} \quad .....(10)$$

The force per unit length of conductor RS is given by-

$$F/l = \frac{\mu_0}{4\pi}\frac{2i_1i_2}{r} \quad .....(11)$$

By Fleming's left hand rule, the direction of this force is towards conductor PQ if i_2 is flowing in the same direction as i_1(Figure 4). Similarly, the force per unit length of conductor PQ due to current i_2 in conductor RS will be same i.e. $F/l = \frac{\mu_0}{4\pi}\frac{2i_1i_2}{r}$ and is directed towards conductor RS . Thus, if the currents are in the same direction, then the nature of the force is attractive. The two conductors will have a tendency to move towards each other. If the two ends of the conductors are fixed, then the shape of two conductors will be concave.

If the direction of currents in two conductors is opposite, the force on two conductors will be outwards i.e. repulsive in nature (Figure 5) and now the conductors will repel to each other. If the ends of two conductors are fixed, then the shape of these conductors will be convex.

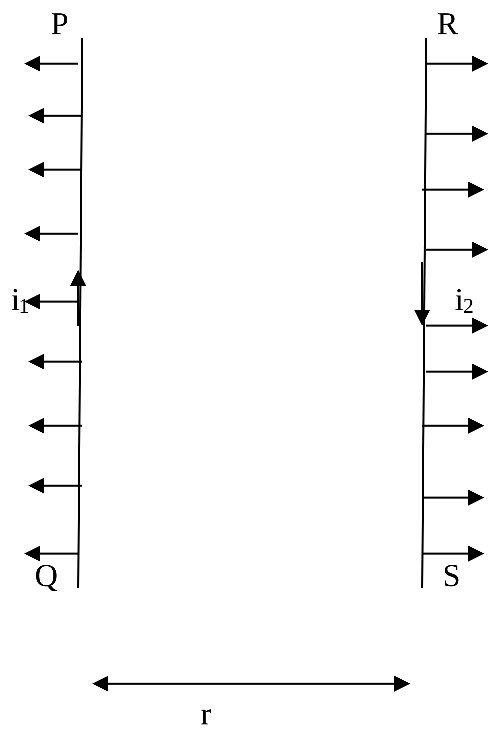

Figure 5

11.5.1 Definition of Ampere:

The force of attraction or repulsion between two long, parallel and straight conductors in vacuum has been used to define ampere.

$$F/l = \frac{\mu_0}{4\pi}\frac{2i_1i_2}{r}$$

Let $i_1 = i_2 = 1$ Amp. and r = 1 meter, then

$$F/l = \frac{\mu_0}{4\pi}\frac{2i^2}{r} = 1 \times 10^{-7} \times \frac{2\times(1)^2}{1}$$

$= 2 \times 10^{-7}$ N/meter

Thus, 1 ampere is the current which when flowing in each of two infinitely long parallel conductors 1 meter apart in vacuum produces between them a force of exactly 2×10^{-7} N/meter of length.

Example 3: Estimate the force per unit length on a long straight wire carrying a current of 4 Amp due to a parallel wire carrying a current of 6 amp. If the direction of currents in two wires is same, then find the nature of force acting between them. The distance between the wires is 3 cm.

Solution: Given $i_1 = 4$ amp, $i_2 = 6$ amp, $r = 3$ cm $= 3 \times 10^{-2}$ m

Using formula $F/l = \frac{\mu_0}{4\pi}\frac{2i_1i_2}{r}$, we get-

Force per unit length $F/l = 1\times 10^{-7}\times\frac{2\times4\times6}{3\times10^{-2}}$

$= 1.6\times10^{-4}$ N/m^{-1}

Since the direction of currents in two wires is same, therefore the force acting between them is attractive in nature.

Example 4: Two parallel wires, 1 m apart, carry currents of 1 amp and 3 amp in opposite directions. Calculate the magnitude and nature of force acting between them on a length of 2 m.

Solution: Given $r = 1$ m, $i_1 = 1$ amp, $i_2 = 3$ amp, $l = 2$ m

Using $F = \frac{\mu_0}{4\pi}\frac{2i_1i_2l}{r}$, we get-

$F = 1\times 10^{-7}\times\frac{2\times1\times3}{1}\times2$

$= 12\times10^{-7}$ N/m (repulsive i.e. away from each other)

Self Assessment Question (SAQ) 6: The parallel wires each of length 200 cm and carrying a current of 0.4 amp in the same direction, are kept 40 cm apart in air. Find the force per unit length on each wire.

Self Assessment Question (SAQ) 7:"Two parallel wires carrying current in the same direction repel each other". Is this statement true or false? Give reason.

11.6 AMPERE'S CIRCUITAL LAW

According to Ampere's circuital law, "The line integral of magnetic induction around a closed path is equal to μ_0 times the net current enclosed by the path" i.e.

$$\oint \vec{B}.\vec{dl} = \mu_0\, i \qquad .....(12)$$

where i is the current enclosed by the path.

Let us suppose that the magnetic field induction B arises due to a long wire carrying a current of i ampere. Now let us consider a circular path of radius r centred on this current carrying wire.

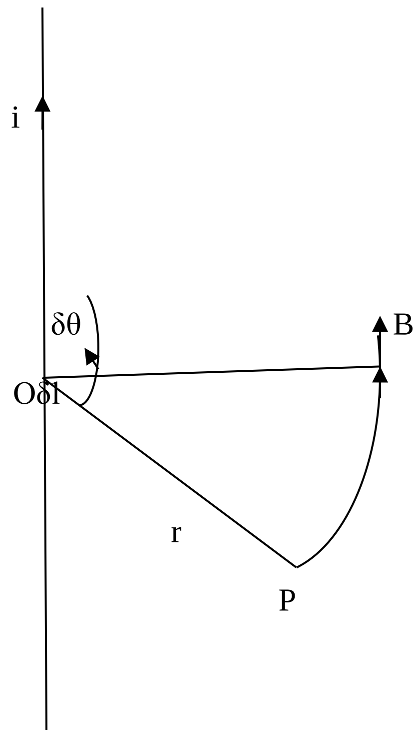

Figure 6

The magnitude of magnetic induction at any point P on the circular path is given by-

$$B = \frac{\mu_0}{4\pi}\frac{2i}{r} \qquad .....(13)$$

For all points on the circular path, the magnetic induction $\vec{B}$ has the same magnitude given by equation (13) and it is parallel to the tangent to the circular path. Therefore, the line integral of the magnetic induction B around the circular pathcentred on the current carrying wire is given by-

$$\oint \vec{B}\,dl = \oint \vec{B}dl = \oint \frac{\mu_0}{4\pi}\frac{2i}{r} r d\theta$$

$$= \frac{\mu_0}{4\pi} 2i \oint \delta\theta$$

$$= \frac{\mu_0}{4\pi} 2i\,(2\pi) = \mu_0\, i$$

Thus we have- $\oint \vec{B}.\vec{dl} = \mu_0\, i$

The sign of integral depends upon the direction in which the current is enriched. The sign is positive if the path followed for line integral is parallel to B and negative if the path followed is anti-parallel.

If the path enclosing the current is not circular but is irregular of any shape, then we divide the path into large number of small elements. Ampere's law holds for closed path of any shape.

11.6.1 Differential form of Ampere's Law

Ampere's circuital law can be expressed in terms of magnetic field intensity ($\vec{H}$). We know that-

$\vec{B} = \mu_0\vec{H}$

Therefore from equation (12) we have-

$\oint \mu_0 \vec{H}.\vec{dl} = \mu_0\, i$

Or $$\oint \vec{H}.\vec{dl} = i \qquad .....(14)$$

But current $i = \iint \vec{J}.\vec{dS}$ .....(15)

Where $\vec{J}$ is the current density and $\vec{dS}$ is small element of area at the point of current density $\vec{J}$ inside the closed path.

Therefore, equation takes the form as-

$\oint \vec{H}.\vec{dl} = \iint \vec{J}.\vec{dS}$ .....(16)

Using Stoke's theorem, we have-

$\oint \vec{H}.\vec{dl} = \iint \text{curl}\, \vec{H}.\vec{dS}$

Therefore, equation (16) becomes-

$\iint \text{curl}\, \vec{H}.\vec{dS} = \iint \vec{J}.\vec{dS}$

i.e. $$\iint (\text{curl}\vec{H} - \vec{J}).\vec{dS} = 0 \qquad .....(17)$$

As the surface is arbitrary, therefore integrand must vanish i.e.

$$\text{curl}\, \vec{H} - \vec{J} = 0$$

or $$\text{curl}\, \vec{H} = \vec{J} \qquad .....(18)$$

Multiplying both sides by μ_0 in equation (18), we get-

$\mu_0 \text{curl}\, \vec{H} = \mu_0\vec{J}$

or $$\text{curl}\, \mu_0\vec{H} = \mu_0\vec{J}$$

or $$\text{curl } \vec{B} = \mu_0 \vec{J} \qquad .....(19)$$

Equation (18) or (19) is the differential form of Ampere's circuital law. The above relation (19) indicates that the magnetic induction at a point is derived from the given value of $\vec{J}$ at that point by integration. However this equation is not enough to derive $\vec{B}$ at a point because for the same value of $\vec{J}$ at the point another term may be added to $\vec{B}$. We, therefore, need another condition.

11.6.2 Applications of Ampere's Law

Magnetic Field due to Long Straight Current Carrying Wire

Let us consider a long straight wire carrying a current i. From the symmetry of wire, it is clear that the magnetic lines of force are concentric circles centred on the wire

$\vec{dl}$

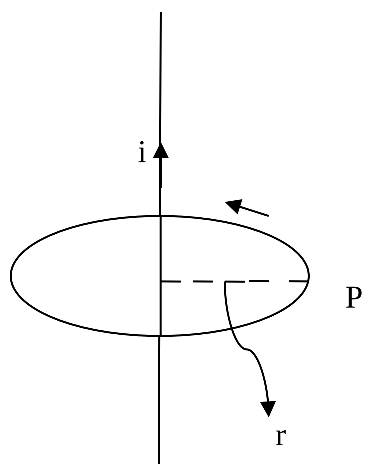

Figure 7

Let P be a point at distance r from the wire at which magnetic field is to be required. Let us consider a circular path of radius r passing through P. By symmetry, the value of magnetic field is same at each point on the circular path. $\vec{B}$ and $\vec{dl}$ are always directed along the same direction. Therefore, the line integral of $\vec{B}$ along the boundary of circular path is-

$$\oint \vec{B}.\vec{dl} = \int B dl \cos 0^0 = B \int dl = B\,(2\pi r)$$

Using Ampere's circuital law-

$$\oint \vec{B}.\vec{dl} = \mu_0\, i$$

Putting for $\oint \vec{B}.\vec{dl}$, we get-

$$B\,(2\pi r) = \mu_0\, i$$

Or $$B = \frac{\mu_0}{2\pi}\frac{i}{r}$$

Or $$B = \frac{\mu_0}{4\pi}\frac{2i}{r}$$

This is the required magnetic field.

11.7 MAXWELL CORRECTION IN AMPERE'S LAW

Let us examine the validity of this equation for time varying fields. Since divergence of curl of any vector quantity is always zero, therefore div curl $\vec{H} = 0$. Then equation (18) curl $\vec{H}=\vec{J}$ implies-

$$\text{div}\,\vec{J} = 0 \qquad .....(20)$$

We knowthe equation of continuity-

$$\text{div}\,\vec{J} + \frac{\partial\rho}{\partial t} = 0 \qquad .....(21)$$

or $$\text{div}\,\vec{J} = -\frac{\partial\rho}{\partial t} \qquad .....(22)$$

Here ρ is the charge density.

From equations (20) and (22), we get-

$$\frac{\partial\rho}{\partial t} = 0$$

or $$\rho = \text{constant}$$

i.e. charge density is static. Thus we conclude that Ampere's circuital law $\oint \vec{H}.\vec{dl} = i$ is valid only for steady state conditions and is insufficient for the cases of time varying fields. Hence to include time varying fields, Ampere's law must be modified. Maxwell investigated mathematically how one could alter Ampere's equation $\oint \vec{H}.\vec{dl} = i$ so as to make it consistent with the equation of continuity.

Maxwell assumed that the definition of current density $\vec{J}$ is incompleteand hence something say $\vec{J_d}$must be added to it.Thus, the total current density, which must be solenoidal, becomesequal to $\vec{J}$ + $\vec{J_d}$. Using this assumption, equation (18) curl $\vec{H}=\vec{J}$becomes-

$$\text{curl}\,\vec{H}=\vec{J} + \vec{J_d} \qquad .....(23)$$

Now let us identify $\vec{J_d}$. Let us take divergence of equation (23) as-

$$\text{div curl } \vec{H} = \text{div } (\vec{J} + \vec{J_d}) \qquad .....(24)$$

But we know that the divergence of curl of any vector quantity is always zeroi.e. div curl $\vec{H}$ = 0, therefore, the above equation takes the form as-

$$\text{div } (\vec{J} + \vec{J_d}) = 0$$

or
$$\text{div } \vec{J} + \text{div } \vec{J_d} = 0$$

or
$$\text{div } \vec{J_d} = - \text{div } \vec{J} \qquad .....(25)$$

We know the equation of continuity-

$$\text{div } \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

or
$$\text{div } \vec{J} = - \frac{\partial \rho}{\partial t}$$

Putting for div $\vec{J}$ in equation (25), we get-

$$\text{div } \vec{J_d} = \frac{\partial \rho}{\partial t} \qquad .....(26)$$

But by differential form of Gauss theorem we have-

$$\text{div } \vec{D} = \rho \qquad .....(27)$$

where $\vec{D}$ is electric displacement vector.

Using equation (27) in equation (26), we get-

$$\text{div } \vec{J_d} = \frac{\partial(\text{div } \vec{D})}{\partial t}$$

$$= \text{div } \left(\frac{\partial \vec{D}}{\partial t}\right)$$

or
$$\vec{J_d} = \left(\frac{\partial \vec{D}}{\partial t}\right) \qquad .....(28)$$

Therefore, the modified form of Ampere's law becomes-

$$\text{curl } \vec{H} = \vec{J} + \vec{J_d} = \vec{J} + \left(\frac{\partial \vec{D}}{\partial t}\right) \qquad .....(29)$$

The additional term which Maxwell added in Ampere's circuital law to include time varying fields is called 'displacement current' because it arises when electric displacement vector $\vec{D}$ changes with time. By the addition of this term Maxwell assumed that this term i.e. displacement current is as effective as the conduction current $\vec{J}$ for producing magnetic field.

Characteristics of displacement current

(a) Displacement current is a current only in the sense that it produces a magnetic field. It has none of the other properties of current since it is not related with the motion of a charge.
(b) Displacement current has a finite value even in a perfect vacuum where there is no charge at all.
(c) The magnitude of displacement current is equal to the rate of change of electric displacement vector i.e. $\vec{J_d} = \left(\frac{\partial \vec{D}}{\partial t}\right)$
(d) Displacement current in a good conductor is negligible as compared to the conduction current at any frequency less than optical frequencies.

Example 5: A 50 V voltage generator at 20 MHz is connected to the plates of air dielectric parallel plate capacitor with plate area 2.8 cm^2 and distance of separation is 0.02 cm. Find the maximum value of displacement current density and displacement current.

Solution: V_o = 50 Volt, f = 20 MHz = 20×10^6 Hz, S = 2.8 cm^2 = 2.8×10^{-4} m^2 , d = 0.02 cm = 2×10^{-4} m

$V = V_o \sin \omega t = V_o \sin 2\pi f t = 50 \sin (2\pi\times20\times10^6 t)$

Displacement current density $\vec{J_d} = \left(\frac{\partial \vec{D}}{\partial t}\right)$

$$= \frac{\partial(\varepsilon_0 \vec{E})}{\partial t} = \frac{\partial}{\partial t}\left(\varepsilon_0 \frac{V}{d}\right)$$

$$= \frac{\varepsilon_0}{d}\frac{\partial V}{\partial t}$$

$$= \frac{\varepsilon_0}{d}\frac{\partial\{50 \sin (2\pi\times20\times10^6 t)\}}{\partial t}$$

$$= \frac{\varepsilon_0}{d}\{50\cos (2\pi \times 20 \times 10^6 t)\}\times 2\pi \times 20 \times 10^6$$

$$= \frac{8.85\times10^{-12}}{2\times10^{-4}}\{50 \cos(2\pi \times 20 \times 10^6 t)\}\times 2\pi \times 20 \times 10^6$$

$= 277.8 \cos (4\pi\times10^7 t)$ Amp/m^2

Displacement current $i_d = J_d\times S = 277.8 \cos (4\pi\times10^7 t)\times2.8\times10^{-4}$

$= 0.0778\times2.8 \cos(4\pi\times10^7 t)$ Amp

Self Assessment Question (SAQ) 8: Choose the correct option-

The concept of displacement current was proposed by-

(i) Faraday (ii) Gauss (iii) Ampere (iv) Maxwell

Self Assessment Question (SAQ) 9: Choose the correct option-

Maxwell's modified Ampere's law is valid-

(i) only when electric field does not change (ii) only when electric field varies with time

(iii) in both of the above situations (iv) none of these

Self Assessment Question (SAQ) 10: Choose the correct option-

The displacement current arises due to-

(i) negative charges only (ii) positive charges only (iii) both negative and positive charges (iv) time varying electric field

Self Assessment Question (SAQ) 11: Choose the correct option-

Displacement current goes through the gap between the plates of a capacitor when the charge of a capacitor is-

(i) zero (ii) decreasing (iii) increasing (iv) remaining constant

Self Assessment Question (SAQ) 12: Choose the correct option-

Displacement current is a current only in the sense that-

(i) it produces a magnetic field (ii) it produces electric field (iii) it produces both fields (iv) none of these

11.8 SUMMARY

In this unit, you have studied about Lorentz force and Biot-Savart law. You have studied that a current carrying conductor produces magnetic field around it. You have also studied about the magnetic force between two current carrying conductors and established its expression and deduced the definition of ampere. You have seen that the conductors attract each other if currents in them are in the same direction and repel each other if currents are in opposite directions. In this unit, you have studied and analyzed Ampere's circuital law and Maxwell's correction in it. According to Ampere's circuital law, the line integral of magnetic induction around a closed path is equal to μ_0 times the net current enclosed by the path. You have also seen that Ampere's law holds for closed path of any shape. You have known about displacement current and its peculiar characteristics. To present the clear understanding and to make the concepts of the unit clear, many solved examples are given in the unit. To check your progress, self assessment questions (SAQs) are given place to place.

11.9 GLOSSARY

Magnetic field- the region surrounding a magnetic

Magnetic induction- a vector which specifies the magnitude and direction of magnetic field at a point

Simultaneous – concurrent, coincident

Electric force- the force experienced by a charge placed at a point in an electric field

Magnetic force- the force experienced by a charge in a magnetic field

Infinitesimal- minute, tiny

Vacuum- emptiness, vacuity

Characteristics- features, qualities

11.10 TERMINAL QUESTIONS

1. Explain the magnitude and direction of the force acting on a charge moving in a magnetic field. When is the force maximum and when minimum?

2. Explain Biot Savart law.

3. Establish the expression for magnetic force acting between two long, parallel and straight current carrying conductors.

4. Both the electric and magnetic field can deflect an electron. What is the difference between these deflections?

5. Explain Ampere's circuital law. Give its significance. Derive its differential form.

6. Explain Maxwell's correction in Ampere's circuital law.

7. Explain the concept of Maxwell's displacement current and show how it led to the modification of the Ampere's law.

8. Obtain the generalized form of Ampere's circuital law. Comment on the concept of the displacement.

9. Throw the light on characteristics of displacement current.

10. Using Ampere's circuital law, establish the expression of magnetic field due to a long current carrying wire.

11. Give a comparison between Coulomb's law and Biot-Savart law.

11.11 ANSWERS

Self Assessment Questions (SAQs):

1. Given $v = 2\times10^8$ m/sec, $B = 0.50$ Wb/m^2 , $q = e = 1.6\times10^{-19}$ C, $m= 9\times10^{-31}$ Kg

Using $F = qvB \sin\theta$, we get-

$F = 1.6\times10^{-19}\times2\times10^8\times0.50\times\sin 90^0 = 1.6\times10^{-11}$ N (towards north, Using Fleming's left hand rule)

Using $F = ma$

Or $a = F/m = 1.6\times10^{-11}/ 9\times10^{-31} = 1.8 \times10^{19}$ m/sec^2

2. Using $F = qvB \sin\theta = evB \sin 90^0 = evB$

Using Fleming's left hand rule, the direction of the force is along –z- axis.

3. Given $K = 2\text{MeV} = 2\times10^6\times1.6\times10^{-19}= 3.2\times10^{-13}$ J, B = 2.5 T, $m = 1.65\times10^{-27}$ Kg

$K = \frac{1}{2} mv^2$ or $v = \sqrt{\frac{2K}{m}} = \sqrt{\frac{2\times3.2\times10^{-13}}{1.65\times10^{-27}}} = 6.23\times10^4$ m/sec^2

Using $F = qvB \sin\theta = 1.6\times10^{-19}\times6.23\times10^4\times2.5\times\sin 90^0 = 7.88\times10^{-12}$ N

4. (iv) 90^0

5. (iii) both electric and magnetic fields

6. Given $l = 200$cm = 2 m, $i_1 = i_2 = 0.4$ amp, r = 40 cm = 0.4 m

$F/l = \frac{\mu_0}{4\pi}\frac{2i_1i_2}{r} = 1\times 10^{-7}\times\frac{2\times0.4\times0.4}{0.4} = 8\times10^{-8}$ N/m (attractive)

7. The statement is false because one current carrying wire will experience force of attraction due to the magnetic field produced by the other current carrying wire.

8. (iv) Maxwell

9. (iii) in both of the above situations

10. (iv) time varying electric field

11. (ii) decreasing (iii) increasing

12. (i) it produces a magnetic field

Terminal Questions:

4. The force exerted by a magnetic field on a moving charge is perpendicular to the motion of the charge; hence the work done by this force on the charge is zero and therefore the kinetic energy of the charge does not change. In an electric field the deflection is in the direction of the field, hence the kinetic energy changes.

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UNIT 12 SEMICONDUCTOR AND SEMICONDUCTOR DEVICES

Structure

12.1 INTRODUCTION

Devices in which a controlled flow of electrons can be obtained are the basic building blocks of all the electronic circuits. Before the discovery of transistor in 1948, such devices were mostly vacuum tubes (also called vacuum valves) like the vacuum diode which has two electrodes, viz., anode (often called plate) and cathode; triode which has three electrodes – cathode, plate and grid; tetrode and pentode (respectively with 4 and 5 electrodes). In a vacuum tube, the electrons are supplied by a heated cathode and the controlled flow of these electrons in vacuum is obtained by varying the voltage between its different electrodes. Vacuum is required in the inter-electrode space; otherwise the moving electrons may lose their energy on collision with the air molecules in their path. In these devices the electrons can flow only from the cathode to the anode (i.e., only in one direction). Therefore, such devices are generally referred to as valves. These vacuum tube devices are bulky, consume high power, operate generally at high voltages (~100 V) and have limited life and low reliability. The seed of the development of modern solid-state semiconductor electronics goes back to 1930's when it was realized that some solid-state semiconductors and their junctions offer the possibility of controlling the number and the direction of flow of charge carriers through them. Simple excitations like light, heat or small applied voltage can change the number of mobile charges in a semiconductor. Note that the supply and flow of charge carriers in the semiconductor devices are within the solid itself, while in the earlier vacuum tubes/valves, the mobile electrons were obtained from a heated cathode and they were made to flow in an evacuated space or vacuum. No external heating or large evacuated space is required by the semiconductor devices.

12.2 OBJECTIVES

After studying this unit, you should be able to-

- Get definition of semiconductor
- Understand different types of semiconductor
- Know various energy bands and charge carriers in semiconductor
- Understand working, characteristic and function of p-n junction diode

12.3. CLASSIFICATION OF METALS, CONDUCTORS AND SEMICONDUCTORS

On the basis of the relative values of electrical conductivity (σ) or resistivity ($\rho = 1/\sigma$), the solids are broadly classified as: (i) Metals: They possess very low resistivity (or high conductivity). $\rho \sim 10^{-2} - 10^{-8}\ \Omega$ m and $\sigma \sim 10^{2} - 10^{8}\ \Omega^{-1}\ m^{-1}$ (ii) Semiconductors: They have resistivity or conductivity intermediate to metals and insulators. $\rho \sim 10^{-5} - 10^{6}\ \Omega$ m and $\sigma \sim 10^{5} - 10^{-6}\ \Omega^{-1}\ m^{-1}$ (iii) Insulators: They have high resistivity (or low conductivity). $\rho \sim 10^{11} - 10^{19}\ \Omega$ m and $\sigma \sim 10^{-}$

$^{11} - 10^{-19}$ Ω^{-1} m^{-1}. The values of ρ and σ given above are indicative of magnitude and could well go outside the ranges as well. Our interest in this chapter is in the study of semiconductors, which could be: (i) Elemental semiconductors: Si and Ge (ii) Compound semiconductors: Examples are: (a) Inorganic: CdS, GaAs, CdSe, InP, etc. (b) Organic: anthracene, doped pthalocyanines, etc. In this chapter, we will restrict ourselves to the study of inorganic semiconductors, particularly elemental semiconductors Si and Ge. The general concepts introduced here for discussing the elemental semiconductors, by-and-large, apply to most of the compound semiconductors as well.

12.4 WHAT ARE SEMICONDUCTORS?

Semiconductors are a group of materials having conductivities between those of metals and insulators. Two general classifications of semiconductors are the elemental semiconductor materials, found in group IV of the periodic table, and the compound semiconductor materials, most of which are formed from special combinations of group III and group V elements.

12.5 TYPES OF SEMICONDUCTORS

On the basis of the absence or presence of impurity atoms the semiconducting materials may be classified as intrinsic or extrinsic semiconductors respectively.

12.5.1 Intrinsic Semiconductors

To understand why Si, Ge, and GaAs are the semiconductors of choice for the electronics industry requires some understanding of the atomic structure of each and how the atoms are bound together to form a crystalline structure. The fundamental components of an atom are the electron, proton, and neutron. In the lattice structure, neutrons and protons form the nucleus and electrons appear in fixed orbits around the nucleus.

12.5.2 Extrinsic Semiconductors

The conductivity of an intrinsic semiconductor depends on its temperature, but at room temperature its conductivity is very low. As such, no important electronic devices can be developed using these semiconductors. Hence there is a necessity of improving their conductivity. This can be done by making use of impurities.

When a small amount, say, a few parts per million (ppm), of a suitable impurity is added to the pure semiconductor, the conductivity of the semiconductor is increased manifold. Such materials are known as e*xtrinsic semiconductors* or *impurity semiconductors*. The deliberate addition of a desirable impurity is called *doping* and the impurity atoms are called *dopants*. Such a material is also called a *doped semiconductor*. There are two extrinsic materials of immeasurable importance to semiconductor device fabrication: *n* -type and *p* -type materials.

12.5.3 Types of Extrinsic Semiconductors

(a) n - type semiconductor

An *n* -type material is created by introducing impurity elements that have *five* valence electrons (*pentavalent*), such as *antimony, arsenic,* and *phosphorus.* Each is a member of a subset group of elements in the Periodic Table of Elements referred to as Group V because each has five valence electrons.

Thus, the pentavalent dopant is donating one extra electron for conduction and hence is known as *donor* impurity. Since the inserted impurity atom has donated a relatively "free" electron to the structure: Diffused impurities with five valence electrons are called donor atoms. It is important to realize that even though a large number of free carriers have been established in the *n* -type material, it is still electrically *neutral* since ideally the number of positively charged protons in the nuclei is still equal to the number of free and orbiting negatively charged electrons in the structure. In a doped semiconductor the total number of conduction electrons n_e is due to the electrons contributed by donors and those generated intrinsically, while the total number of holes n_h is only due to the holes from the intrinsic source. But the rate of recombination of holes would increase due to the increase in the number of electrons. As a result, the number of holes would get reduced further. Thus, with proper level of doping the number of conduction electrons can be made much larger than the number of holes. These semiconductors are, therefore, known as n-*type semiconductors*. For n-type semiconductors, we have, $n_e >> n_h$.

(b) p - type Semiconductor

The *p* -type material is formed by doping a pure germanium or silicon crystal with impurity atoms having *three* valence electrons. The elements most frequently used for this purpose are *boron, gallium,* and *indium*. Each is a member of a subset group of elements in the Periodic Table of Elements referred to as Group III because each has three valence electrons.

Note that there is now an insufficient number of electrons to complete the covalent bonds of the newly formed lattice. The resulting vacancy is called a *hole* and is represented by a small circle or a plus sign, indicating the absence of a negative charge. Since the resulting vacancy will readily *accept* a free electron: The diffused impurities with three valence electrons are called acceptor atoms. The resulting *p* -type material is electrically neutral, for the same reasons described for the *n* -type material.

It is obvious that one *acceptor* atom gives one *hole*. These holes are in addition to the intrinsically generated holes while the source of conduction electrons is only intrinsic generation. For p-type semiconductors, the recombination process will reduce the number (n_i)of intrinsically generated electrons to n_e. We have, for p-type semiconductors n_h >> n_e.

For p-type semiconductor, the acceptor energy level E_A is slightly above the top E_V of the valence band as shown in Figure 8. With very small supply of energy an electron from the valence band can jump to the level E_A and ionize the acceptor negatively. (Alternately, we can also say that with very small supply of energy the hole from level E_A sinks down into the valence band. Electrons rise up and holes fall down when they gain external energy.) At room temperature, most of the acceptor atoms get ionized leaving holes in the valence band. Thus at room temperature the density of holes in the valence band is predominantly due to impurity in the extrinsic semiconductor.

12.6 p-n JUNCTIONS

When a p-type semiconductor is suitably joined to n-type semiconductor, the contact surface is called **p-n junction.** Most semiconductor devices contain one or more *p-n* junctions. The *p-n* junction is of great importance because it is in effect, the *control element* for semiconductor devices. A thorough knowledge of the formation and properties of *p-n* junction can enable the reader to understand the semiconductor devices. A clear understanding of the junction behavior is important to analyse the working of other semiconductor devices. We will now try to understand how a junction is formed and how the junction behaves under the influence of external applied voltage.

12.7 p-n JUNCTION FABRICATION AND PROPERTIES

In actual practice, the characteristic properties of *p-n* junction will not be apparent if a *p*-type block is just brought in contact with *n*-type block. In fact, *p-n* junction is fabricated by special techniques. One common method of making *p-n* junction is called *alloying*. In this method, a small block of indium (trivalent impurity) is placed on an *n*-type germanium slab as shown in Figure (15 *i*). The system is then heated to a temperature of about 500ºC. The indium and some of the germanium melt to form a small puddle of molten germanium-indium mixture as shown in Figure (15 *ii*). The temperature is then lowered and puddle begins to solidify. Under proper conditions, the atoms of indium impurity will be suitably adjusted in the germanium slab to form a single crystal. The addition of indium overcomes the excess of electrons in the *n*-type germanium to such an extent that it creates a *p*-type region. As the process goes on, the remaining molten mixture becomes increasingly rich in indium. When all germanium has been redeposited, the remaining material appears as indium button which is frozen on to the outer surface of the crystallized portion as shown in Figure. This button serves as a suitable base for soldering on leads.

Properties of *p-n* Junction

At the instant of *p-n*-junction formation, the free electrons near the junction in the *n* region begin to diffuse across the junction into the *p* region where they combine with holes near the junction. The result is that *n* region loses free electrons as they diffuse into the junction. This creates a layer of positive charges (pentavalent ions) near the junction. As the electrons move across the junction, the *p* region loses holes as the electrons and holes combine. The result is that there is a layer of negative charges (trivalent ions) near the junction.

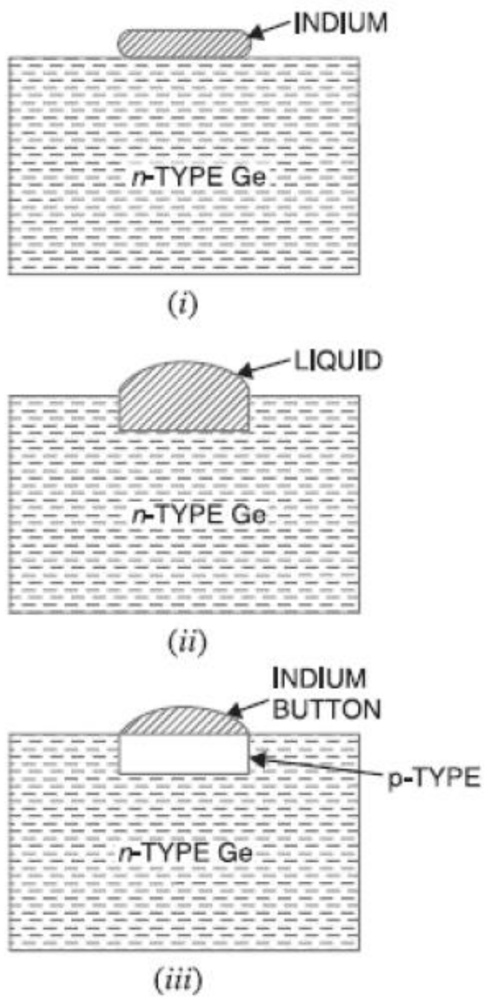


Figure 12.1: Fabrication of p-n junction.

These two layers of positive and negative charges form the *depletion region* (or *depletion layer*). The term depletion is due to the fact that near the junction, the region is depleted (*i.e.* emptied) of *charge carries* (free electrons and holes) due to diffusion across the junction.

It may be noted that depletion layer is formed very quickly and is very thin compared to the *n* region and the *p* region. For clarity, the width of the depletion layer is shown exaggerated.

Once *p-n* junction is formed and depletion layer created, the diffusion of free electrons stops. In other words, the depletion region acts as a barrier to the further movement of free electrons across the junction. The positive and negative charges set up an electric field.

The electric field is a barrier to the free electrons in the *n*-region. There exists a potential difference across the depletion layer and is called **barrier potential** (V_0). The barrier potential of a *p-n* junction depends upon several factors including the type of semiconductor material, the amount of doping and temperature. The typical barrier potential is approximately: For silicon, V_0 = 0.7 V ; For germanium, V_0 = 0.3 V.

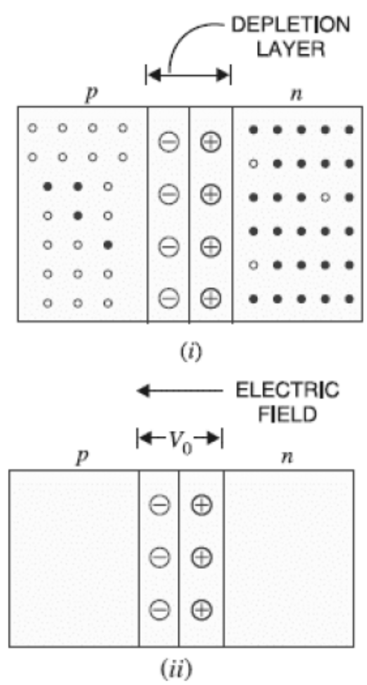


Figure 12.2: p-n junction.

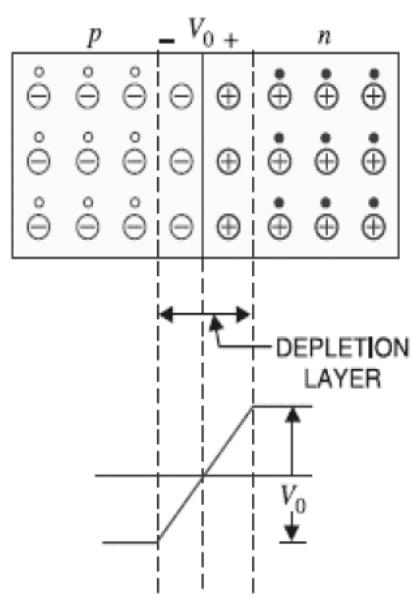


Figure 12.3: Potential distribution curve.

12.8 p-n JUNCTION DIODE

Biasing of p-n junction diode: In electronics, the term bias refers to the use of dc voltage to establish certain operating conditions for an electronic device. In relation to a *p-n* junction, there are following two bias conditions

1. Forward Biasing
2. Reverse Biasing

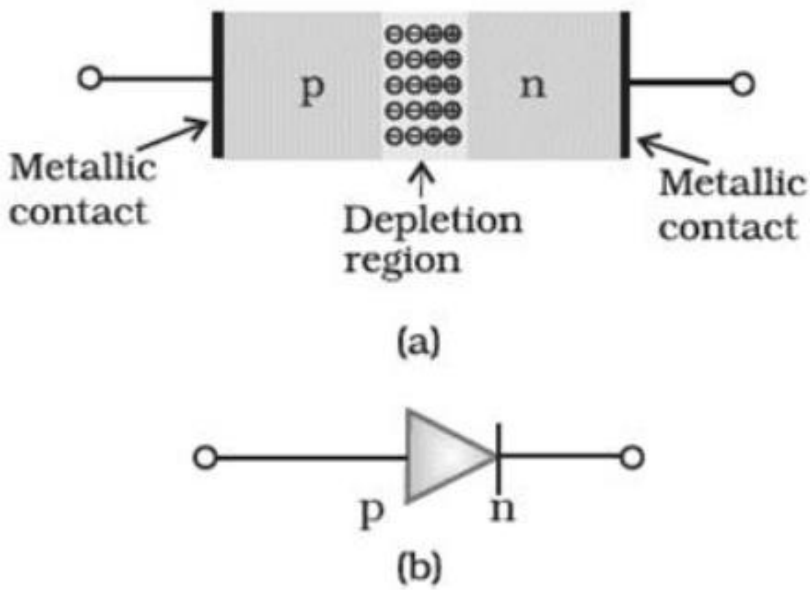


Figure 12.4: (a) Semiconductor diode (b) Symbol for p-n junction diode.

1. Forward Biasing

When external d.c. voltage applied to the junction is in such a direction that it cancels the potential barrier, thus permitting current flow, it is called **forward biasing.**

To apply forward bias, connect positive terminal of the battery to *p*-type and negative terminal to *n*-type as shown in Figure. The applied forward potential establishes an electric field which acts against the field due to potential barrier. Therefore, the resultant field is weakened and the barrier height is reduced at the junction as shown in Figure. As potential barrier voltage is very small (0.1 to 0.3 V), therefore, a small forward voltage is sufficient to completely eliminate the barrier.

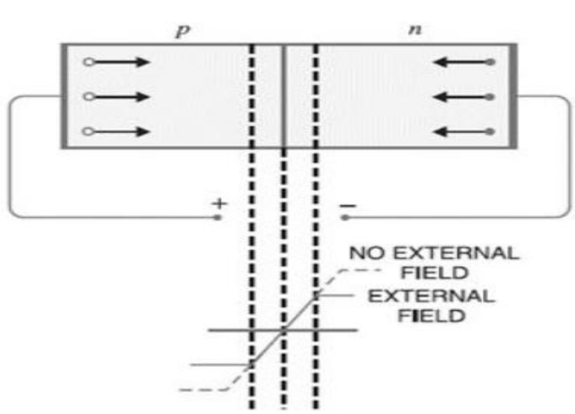


Figure 12.5: Forward Biasing of p-n junction diode.

Once the potential barrier is eliminated by the forward voltage, junction resistance becomes almost zero and a low resistance path is established for the entire circuit. Therefore, current flows in the circuit. This is called *forward current*.

2. Reverse Biasing

When the external d.c. voltage applied to the junction is in such a direction that potential barrier is increased, it is called **reverse biasing.**

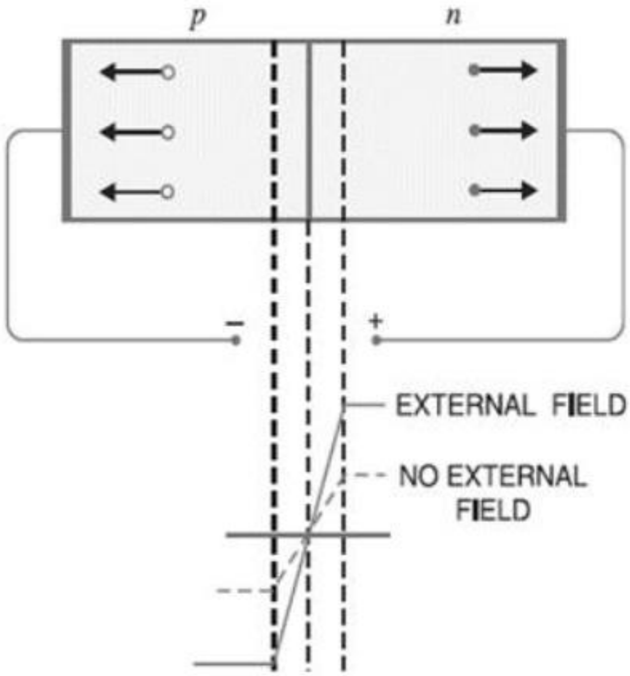


Figure 12.6: Reverse Biasing of p-n junction diode.

To apply reverse bias, connect negative terminal of the battery to *p*-type and positive terminal to *n*-type as shown in Figure. It is clear that applied reverse voltage establishes an electric field which acts in the same direction as the field due to potential barrier. Therefore, the resultant field at the junction is strengthened and the barrier height is increased as shown in Figure. The increased potential barrier prevents the flow of charge carriers across the junction.

12.9. Zener Diode

It is a special purpose semiconductor diode, named after its inventor C. Zener. It is designed to operate under reverse bias in the breakdown region and used as a voltage regulator. The symbol for Zener diode is shown in Figure 27. Zener diode is fabricated by heavily doping both p-, and n-sides of the junction. Due to this, depletion region formed is very thin ($<10^{-6}$m) and the electric field of the junction is extremely high ($\sim 5 \times 10^{6}$ V/m) even for a small reverse bias voltage of about 5V.

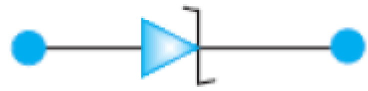


Figure 12.7: Circuit symbol of Zener diode.

The breakdown or zener voltage depends upon the amount of doping. If the diode is heavily doped, depletion layer will be thin and consequently the breakdown of the junction will occur at a lower reverse voltage. On the other hand, a lightly doped diode has a higher breakdown voltage. When an ordinary crystal diode is properly doped so that it has a sharp breakdown voltage, it is called a zener diode. A typical Zener diode characteristic is shown in **Figure**. The maximum reverse current, $I_{Z(\max)}$ which the Zener diode can withstand is dependent on the design and construction of the diode. A design guideline that the minimum Zener current, where the characteristic curve remains at V_z (near the knee of the curve), is $0.1/I_{Z(\max)}$.

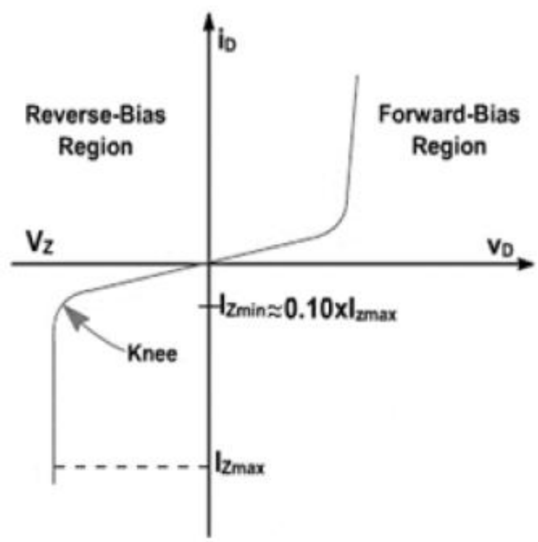


Figure 12. 8: Zener diode characteristics.

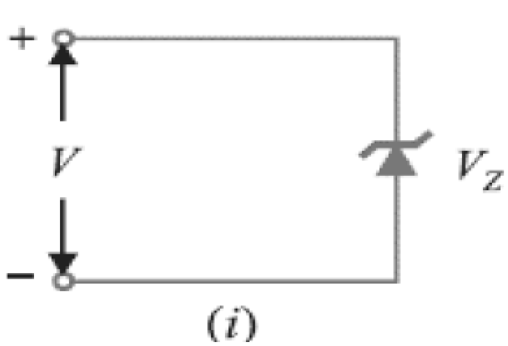


Figure 12.9: V ≥ V_Z

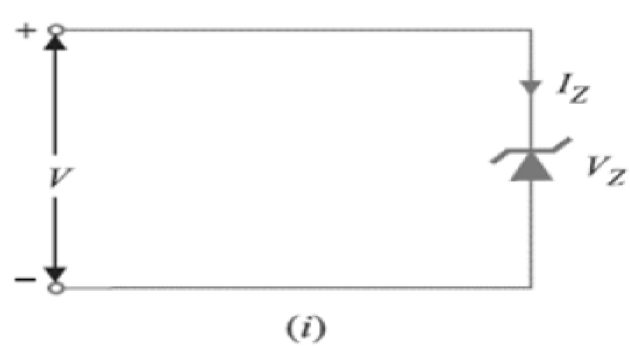
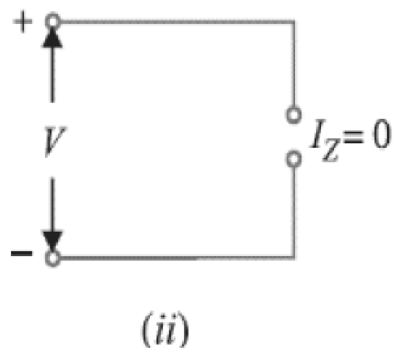


Figure 12.10: Equivalent circuit of Zener for "ON" state.



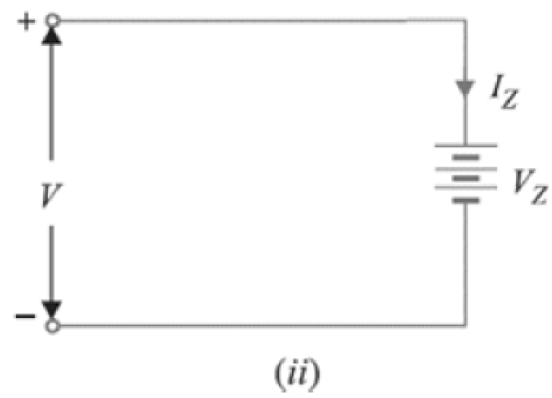


Figure 12.11:1 V_Z > V > 0.

Figure 12.12: Equivalent circuit of Zener for "OFF" state.

The power handling capacity of these diodes is better. The power dissipation of a zener diode equals the product of its voltage and current.

$$P_Z = V_Z I_Z$$

The amount of power which the zener diode can withstand ($V_Z I_{Z(\max)}$) is a limiting factor in power supply design.

Equivalent Circuit of Zener Diode

The analysis of circuits using zener diodes can be made quite easily by replacing the zener diode by its equivalent circuit.

(i) **"On" state**. When reverse voltage across a zener diode is equal to or more than break down voltage V_Z, the current increases very sharply. In this region, the curve is almost vertical. It means that voltage across zener diode is constant at V_Z even though the current through it changes. Therefore, in the breakdown region, an ideal zener diode can be represented by a battery of voltage V_Z as shown in Figure (*i*). Under such conditions, the zener diode is said to be in the "ON" state.

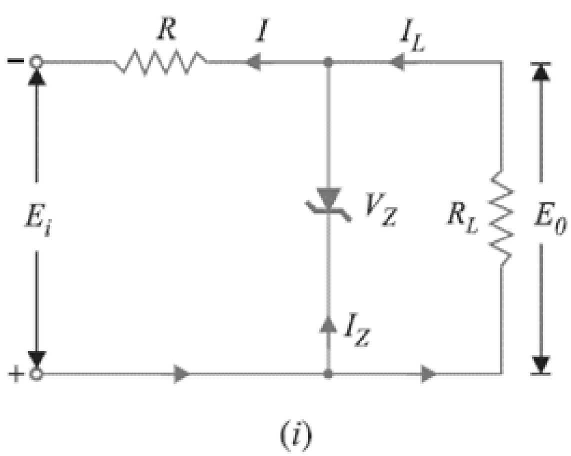
(ii) **"OFF" state.** When the reverse voltage across the zener diode is less than V_Z but greater than 0 V, the zener diode is in the "OFF" state. Under such conditions, the zener diode can be represented by an open-circuit as shown in Fig.32.

Zener Diode as Voltage Stabilizer

A zener diode can be used as a voltage regulator to provide a constant voltage from a source whose voltage may vary over sufficient range. The circuit arrangement is shown in Figure (*i*). The zener diode of zener voltage V_Z is reverse connected across the load R_L across which constant output is desired. The series resistance R absorbs the output voltage fluctuations so as to maintain constant voltage across the load. It may be noted that the zener will maintain a constant voltage V_Z (=E_0) across the

Load so long as the input voltage does not fall below V_Z. When the circuit is properly designed, the load voltage E_0 remains essentially constant (equal to V_Z) even though the input voltage E_i and load resistance R_L may vary over a wide range.

Suppose the input voltage increases. Since the zener is in the breakdown region, the zener diode is equivalent to a battery V_Z as shown in Fig.34. It is clear that output voltage remains constant at V_Z(= E_0). The excess voltage is dropped across the series resistance R.
This will cause an increase in the value of total current I. The zener will conduct the increase of current in I while the load current remains constant. Hence, output voltage E_0 remains constant irrespective of the changes in the input voltage E_i. Now suppose that input voltage is constant but the load resistance R_L decreases. This will cause an increase in load current. The extra current cannot come from the source because drop in R (and hence source current I) will not change as the zener is within its regulating range.



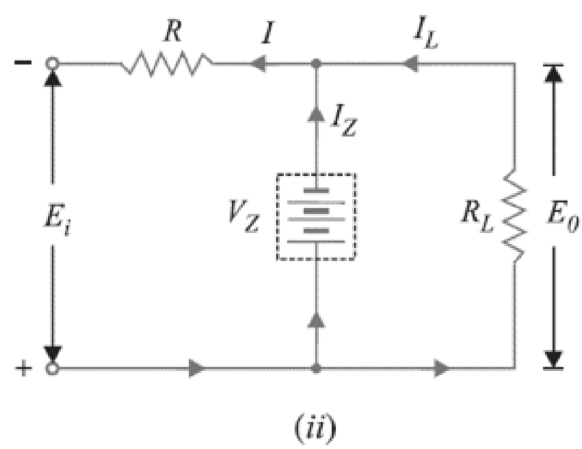


Figure 12.13: Zener as a Voltage Regulator. voltage

Figure 12.14: Equivalent circuit of zener as a regulator.

The additional load current will come from a decrease in Zener current I_Z. Consequently, the output voltage stays at constant value.

Voltage drop across $R = E_i - E_0$

Current through R, $I = I_Z + I_L$

Applying Ohm's law, we have,

$$R = \frac{E_i - E_0}{I_Z + I_L}$$

12.10 TUNNEL DIODE

The tunnel diode was first introduced by Leo Esaki in 1958. Its characteristics, shown in Figure, are different from any diode discussed thus far in that it has a negative-resistance region. In this region, an increase in terminal voltage results in a reduction in diode current.

The tunnel diode is fabricated by doping the semiconductor materials that will form the *p-n* junction at a level 100 to several thousand times that of a typical semiconductor diode. This results in a greatly reduced depletion region, of the order of magnitude of 10^{-6}cm, or typically about 1/100 the width of this region for a typical semiconductor diode. It is this thin depletion region, through which many carriers can "tunnel" rather than attempt to surmount, at low forward-bias potentials that accounts for the peak in the curve of Figure. For comparison purposes, a typical semiconductor diode characteristic is superimposed on the tunnel-diode characteristic of Figure 35.

This reduced depletion region results in carriers "punching through" at velocities that far exceed those available with conventional diodes. The tunnel diode can therefore be used in high-speed applications such as in computers, where switching times in the order of nanoseconds or picoseconds are desirable. We know that an increase in the doping level reduces the Zener potential. Note the effect of a very high doping level on this region in Figure. The semiconductor materials most frequently used in the manufacture of tunnel diodes are germanium and gallium arsenide. The ratio $I_P > I_V$is very important for computer applications. For germanium, it is typically 10:1, and for gallium arsenide, it is closer to 20:1.

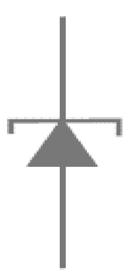


Figure 12.15: Circuit symbol of Tunnel diode.

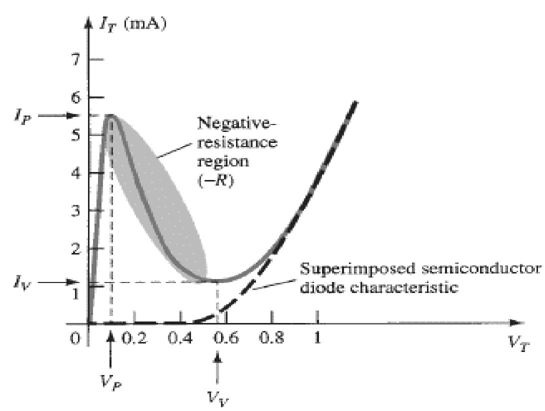


Figure 12.16: Tunnel diode characteristics.

The peak current I_P of a tunnel diode can vary from a few microamperes to several hundred amperes. The peak voltage, however, is limited to about 600 mV. For this reason, a simple VOM with an internal dc battery potential of 1.5 V can severely damage a tunnel diode if applied improperly.

12.11 PHOTO DIODE AND LED

We have seen so far, how a semiconductor diode behaves under applied electrical inputs. Now we learn about semiconductor diodes in which carriers are generated by photons (photo-excitation). All these devices are called *optoelectronic devices*. We shall study the functioning of the following optoelectronic devices:
(i) *Photodiodes* used for detecting optical signal (photo detectors).
(ii) *Light emitting diodes* (LED) which convert electrical energy into light.
(iii) *Photovoltaic devices* which convert optical radiation into electricity (*solar cells*).

(i) Photo Diode

A Photodiode is again a special purpose p-n junction diode fabricated with a transparent window to allow light to fall on the diode. It is operated under reverse bias. When the photodiode is illuminated with light (photons) with energy ($h\nu$) greater than the energy gap (Eg) of the semiconductor, then electron-hole pairs are generated due to the absorption of photons. The diode is fabricated such that the generation of *e-h* pairs takes place in or near the depletion region of the diode. Due to electric field of the junction, electrons and holes are separated before they recombine. The direction of the electric field is such that electrons reach n-side and holes reach p-side. Electrons are collected on n-side and holes are collected on p-side giving rise to an emf.

Principle: When a rectifier diode is reverse biased, it has a very small reverse leakage current. The same is true for a photo-diode. The reverse current is produced by thermally generated electron hole pairs which are swept across the junction by the electric field created by the reverse voltage. In a rectifier diode, the reverse current increases with temperature due to an increase in the number of electron-hole pairs. A photo-diode differs from a rectifier diode in that when its p-n junction is exposed to light, the reverse current increases with the increase in light intensity and vice-versa. This is explained as follows. When light (photons) falls on the p-n junction, the energy is imparted by the photons to the atoms in the junction. This will create more free electrons (and more holes). These additional free electrons will increase the reverse current. As the intensity of light incident on the p-n junction increases, the reverse current also increases. In other words, as the incident light intensity increases, the resistance of the device (photo-diode) decreases.
The circuit diagram used for the measurement of *I-V* characteristics of a photodiode is shown in Fig. 12.17(a) and a typical I-*V* characteristics in Fig. 12.17 (b).

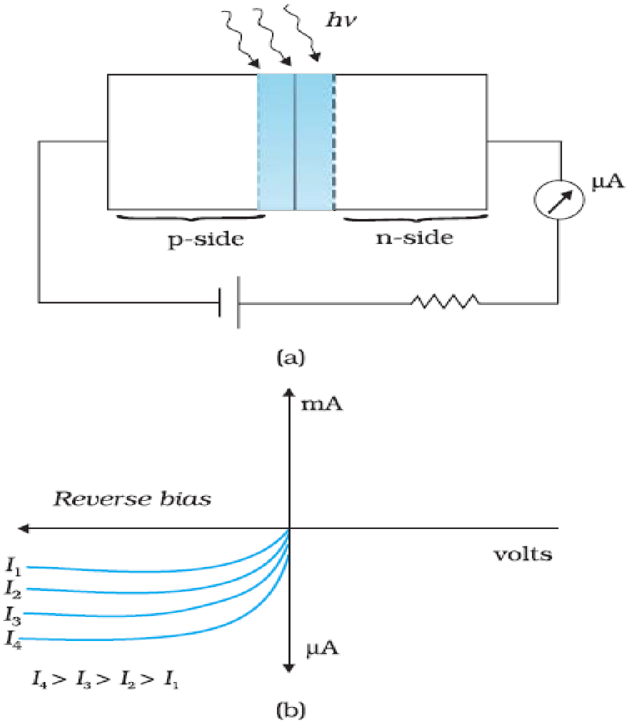


Figure 12.17: (a) An illuminated photodiode under reverse bias, (b) *I-V* characteristics of a Photodiode for different illumination intensity $I_4 > I_3 > I_2 > I_1$.

Photo-diode package. Figure 41 (*i*) shows a typical photo-diode package. It consists of a *p-n* junction mounted on an insulated substrate and sealed inside a metal case. A glass window is mounted on top of the case to allow light to enter and strike the *p-n* junction. The two leads extending from the case are labelled anode and cathode. The cathode is typically identified by a tab extending from the side of the case. Figure 18 (*ii*) shows the schematic symbol of a photo-diode. The inward arrows represent the incoming light.

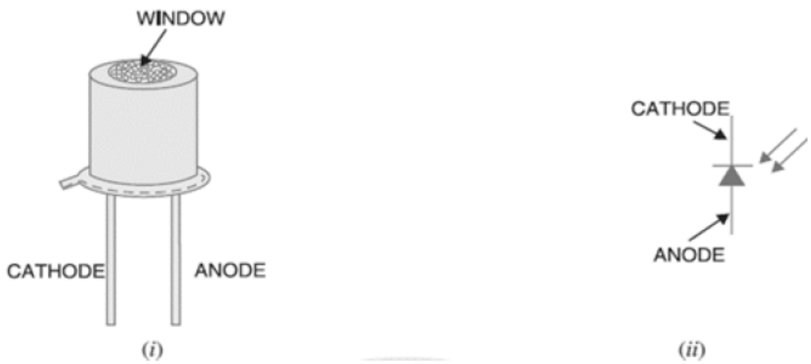


Figure 12.18: Photo diode.

Characteristics of Photo-diode There are two important characteristics of photodiode.

(*i*) Reverse current -Illumination curve. Figure 12.19 shows the graph between reverse current (I_R) and illumination (E) of a photo-diode. The reverse current is shown on the vertical axis and is measured in µA. The illumination is indicated on the horizontal axis and is measured in mW/cm^2. Note that graph is a straight line passing through the origin. $I_R = m\ E$ where m = slope of the straight line. The quantity m is called the *sensitivity* of the photo-diode.

(*ii*) Reverse voltage -Reverse current curve. Figure 12.20 shows the graph between reverse current (I_R) and reverse voltage (V_R) for various illumination levels. It is clear that for a given reverse-biased voltage, the reverse current I_R increases as the illumination (E) on the *p-n* junction of photo-diode are increased.

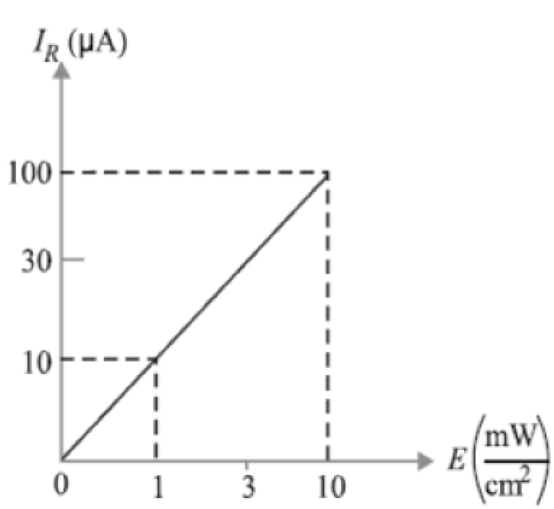


Figure 12.19: Reverse current I_R versus illumination E of photo diode.

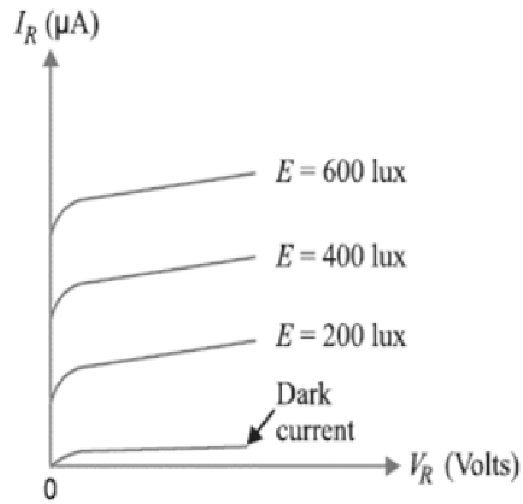


Figure 12.20: Reverse current I_R versus reverse voltage V_R

Applications of Photo diode: Some common applications include alarm system and counter circuits.

(ii) Light Emitting Diode

It is a heavily doped p-n junction which under forward bias emits spontaneous radiation. The diode is encapsulated with a transparent cover so that emitted light can come out. When the diode is forward biased, electrons are sent from n → p (where they are minority carriers) and holes are sent from p → n (where they are minority carriers). At the junction boundary the concentration of minority carriers increases compared to the equilibrium concentration (i.e., when there is no bias). Thus at the junction boundary on either side of the junction, excess minority carriers are there which recombine with majority carriers near the junction. On recombination, the energy is released in the form of photons. Photons with energy equal to or slightly less than the band gap are emitted. When the forward current of the diode is small, the intensity of light emitted is small. As the forward current increases, intensity of light increases and reaches a maximum. Further increase in the forward current results in decrease of light intensity. LEDs are biased such that the light emitting efficiency is maximum. The *V-I* characteristics of a LED is similar to that of a Si junction diode. But the threshold voltages are much higher and slightly different for each colour. The reverse breakdown voltages of LEDs are very low, typically around 5V. So care should be taken that high reverse voltages do not appear across them. LEDs that can emit red, yellow, orange, green and blue light are commercially available. The semiconductor used for fabrication of visible LEDs must at least have a band gap of 1.8 eV (spectral range of visible light is from about 0.4 µm to 0.7 µm, i.e., from about 3 eV to 1.8 eV). The compound semiconductor Gallium Arsenide – Phosphide ($GaAs_{1-x}P_x$) is used for making LEDs of different colours. $GaAs_{0.6}P_{0.4}$ (Eg ~ 1.9 eV) is used for red LED. GaAs (Eg ~ 1.4 eV) is used for making infrared LED. These LEDs find extensive use in remote controls, burglar alarm systems, optical communication, etc. Extensive research is being done for developing white LEDs which can replace incandescent lamps. LEDs have the following advantages over conventional incandescent low power lamps:

(i) *Low operational voltage and less power.*

(ii) *Fast action and no warm-up time required.*

(iii) *The bandwidth of emitted light is 100 Å to 500 Å or in other words it is nearly (but not exactly) monochromatic.*

(iv) *Long life and ruggedness.*

(v) *Fast on-off switching capability.*

The increasing use of digital displays in calculators, watches, and all forms of instrumentation has contributed to an extensive interest in structures that emit light when properly biased. The two types in common use to perform this function are the light-emitting diode (LED) and the liquid-crystal display (LCD). The light-emitting diode is a diode that gives off visible or invisible (infrared) light when energized. In any forward-biased $p - n$ junction there is, within the structure and primarily close to the junction, a recombination of holes and electrons. This recombination requires that the energy possessed by the unbound free electrons be transferred to another state. In all semiconductor $p - n$ junctions some of this energy is given off in the form of heat and some in the form of photons.

Advantages of LED

The light-emitting diode (LED) is a solid-state light source. LEDs have replaced incandescent lamps in many applications because they have the following advantages:

(*i*) Low voltage

(*ii*) Longer life (more than 20 years)

(*iii*) Fast on-off switching

12.12 SUMMARY

Semiconductors are the basic materials used in the present solid state electronic devices like diode, transistor, ICs, etc. Metals have low resistivity (10–2 to 10–8 Ωm), insulators have very high resistivity (>108 Ωm–1), while semiconductors have intermediate values of resistivity. Semiconductors are elemental (Si, Ge) as well as compound (GaAs,CdS, etc.). Pure semiconductors are called 'intrinsic semiconductors'. The presence of charge carriers (electrons and holes) is an 'intrinsic' property of the material and these are obtained as a result of thermal excitation. The number of electrons (n_e) is equal to the number of holes (n_h) in intrinsic conductors. Holes are essentially electron vacancies with an effective positive charge.

In n-type semiconductors, $n_e >> n_h$ while in p-type semiconductors $n_h >> n_e$. n-type semiconducting Si or Ge is obtained by doping with pentavalent atoms (donors) like As, Sb, P, etc., while p-type Si or Ge can be obtained by doping with trivalent atom (acceptors) like B, Al, In etc. For insulators $Eg > 3$ eV, for semiconductors Eg is 0.2 eV to 3 eV, while for metals $Eg \approx 0$. Zener diode is one such special purpose diode. In reverse bias, after a certain voltage, the current suddenly increases (breakdown voltage) in a Zener diode. This property has been used to obtain *voltage regulation*. p-n junctions have also been used to obtain many photonic or optoelectronic devices where one of the participating entity is 'photon':(a) Photodiodes in which photon excitation results in a change of reverse saturation current which helps us to measure light intensity; (b) Solar cells which convert photon energy into electricity; (c) Light Emitting Diode and Diode Laser in which electron excitation by a bias voltage results in the generation of light.

12.13 GLOSSARY

LED: Light emitting diode

12.14 REFERENCES

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12.15 Terminal question

1. The number of silicon atoms per m^3 is 5×10^{28}. This is doped simultaneously with 5×10^{22} atoms per m3 of Arsenic and 5×10^{20} per m^3 atoms of Indium. Calculate the number of electrons and holes. Given that $n_i = 1.5 \times 1016\ m^{-12.}$ Is the material n-type or p-type?
2. In an intrinsic semiconductor the energy gap E_g is 1.2eV. Its hole mobility is much smaller than electron mobility and independent of temperature. What is the ratio between conductivity at 600K and that at 300K? Assume that the temperature dependence of intrinsic carrier concentration n_i is given by

$$n_t = n_0 \exp\left(-\frac{E_g}{2k_B T}\right)$$ where n_0 is a constant.

3. A p-n photodiode is fabricated from a semiconductor with band gap of 2.8 eV. Can it detect a wavelength of 6000 nm?
4. In a p-n junction diode, the current I can be expressed as $I = I_0 \exp\{ (eV/2k_BT) – 1\}$ where I_0 is called the reverse saturation current, V is the voltage across the diode and is positive for forward bias and negative for reverse bias, and I is the current through the diode, k_B is the Boltzmann constant (8.6×10–5 eV/K) and T is the absolute temperature. If for a given diode $I0 = 5 \times 10–12$ A and T = 300 K, then
   (a) What will be the forward current at a forward voltage of 0.6 V?
   (b) What will be the increase in the current if the voltage the diode is increased to 0.7 V?
   (c) What is the dynamic resistance?
   (d) What will be the current if reverse bias voltage changes from 1 V to 2 V?